If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Jan 28.

[05.1] Determine partial fraction expansions for \( \frac{1}{\cos^2 \pi x} \) and \( \tan \pi x \).

[05.2] Determine the product expansion for \( \cos \pi x \).

[05.3] Exhibit a meromorphic function on \( \mathbb{C} \) with simple poles at points \( \log n \) for \( n = 1, 2, 3, 4, 5, \ldots \) and no other poles. Also, contemplate the analogous, considerably more difficult, question when the residue is required to be \( 1 \) at every pole.

[05.4] Check that the Euclidean Laplacian \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \) on \( \mathbb{R}^n \) is rotation-invariant, in the following sense. A rotation is a linear map \( g : \mathbb{R}^n \to \mathbb{R}^n \) preserving the usual inner product \( \langle x, y \rangle = \sum_i x_i y_i \), and preserving orientations (so \( \det g = 1 \), rather than \( -1 \)). The asserted rotation-invariance is

\[
\Delta(f \circ g) = (\Delta f) \circ g
\]

(for twice-differentiable \( f \) and rotation \( g \)).

(In fact, \( \Delta \) is also preserved by reflections, which are orientation-reversing, so the determinant condition can be safely ignored.)

[05.5] Check that for harmonic \( h \) and holomorphic \( f \), the composition \( h \circ f \) is invariably harmonic, while \( f \circ h \) need not be. (Yes, much of the issue is suitable formulation of the computation.)

[05.6] Show that every harmonic function \( u \) on an annulus \( r < |z| < R \) is of the form

\[
u(z) = a_0 + b_0 \log |z| + \sum_{0 \neq n \in \mathbb{Z}} (a_n z^n + b_n \overline{z}^n)
\]

for constants \( a_i, b_i \). (Hint: Separate variables by writing a Fourier expansion in \( \arg z \), with coefficients depending on the radial coordinate.)

[05.7] (Euler-type equations of second order) An ordinary differential equation of the form

\[
u'' + \frac{b}{x} u' + \frac{c}{x^2} u = 0
\]

with constants \( b, c \) is said to be of Euler type. Show that it has solutions \( x^\alpha \) and \( x^\beta \) where \( \alpha, \beta \) are solutions of the auxiliary equation

\[
\lambda(\lambda - 1) + b\lambda + c = 0
\]

Show that \( x^\alpha \log x \) is the second solution if the root of the auxiliary equation is double, i.e., if \( \alpha = \beta \). Use the Mean Value Theorem to genuinely prove that there are no other solutions.

[05.8] (Rotationally invariant harmonic functions in \( \mathbb{R}^n \)) For \( f \) twice-differentiable on \( \mathbb{R}^n \), expressible as a (twice-differentiable) function of the radius \( r \) alone (at least away from 0), say \( f \) is spherically symmetric or rotationally invariant. (This could also be formulated as invariance under the action of the orthogonal group by rotations). Show that

\[
\Delta f = f'' + \frac{n-1}{r} f'
\]
(This is of Euler type). On \( \mathbb{R}^n - \{0\} \), find two linearly independent harmonic functions.

[05.9] Prove that for non-vanishing entire \( f \), the function \( F(z) = \int_{0}^{z} \frac{f'(w)}{f(w)} \, dw \) is entire, and essentially gives a logarithm of \( f \), in the sense that \( f(z) = e^{C+F(z)} \) for a suitable constant \( C \).

[05.10] Define \( f \) on the unit circle by \( f(e^{i\theta}) = \theta^2 \), for \(-\pi < \theta < \pi\). Find a harmonic function \( u \) on the open disk whose boundary values are \( f \).

[05.11] The Fourier expansion

\[
\delta(\theta) = \sum_{n \in \mathbb{Z}} e^{in\theta} = \sum_{n \in \mathbb{Z}} \hat{\delta}(n) e^{in\theta} \quad \text{(with } \hat{\delta}(n) = 1 \text{ for all } n \in \mathbb{Z})
\]

certainly does not converge pointwise, but does make sense as the expansion of the periodic Dirac \( \delta \), sometimes called Dirac comb function on \( \mathbb{R}/2\pi\mathbb{Z} \), in the following sense. The Plancherel identity

\[
\langle u, v \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} u(\theta) \overline{v(\theta)} \, d\theta = \sum_{n \in \mathbb{Z}} \hat{u}(n) \cdot \overline{\hat{v}(n)} \quad \text{(for } u, v \in L^2(S^1))
\]

\( L^2(S^1) \times L^2(S^1) \to \mathbb{C} \) can be restricted in the first argument and extended in the second, so that for smooth \( u(\theta) = \sum_{n \in \mathbb{Z}} \hat{u}(n) e^{in\theta} \), pairing against \( \delta \) correctly evaluates \( u \) at \( \theta = 0 \):

\[
u(0) = \sum_n \hat{u}(n) e^{i\cdot0} = \sum_n \hat{u}(n) \cdot 1 = \sum_n \hat{u}(n) \cdot \hat{\delta}(n) = \langle u, \delta \rangle
\]

Identifying the circle with the boundary \( \{ z : |z| = 1 \} \) of the disk \( \{ z : |z| < 1 \} \), determine the harmonic function on the disk whose boundary value function is the periodic Dirac \( \delta \).