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Complex analysis examples 05

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[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2021-22/cx_ex_05.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Jan 28.

[05.1] Determine partial fraction expansions for $\frac{1}{\cos^2 \pi x}$ and $\tan \pi x$.

[05.2] Determine the product expansion for $\cos \pi x$.

[05.3] Exhibit a meromorphic function on $\mathbb{C}$ with simple poles at points $\log n$ for $n = 1, 2, 3, 4, 5, \ldots$ and no other poles. Also, *contemplate* the analogous, considerably more difficult, question when the residue is required to be 1 at every pole.

[05.4] Check that the Euclidean Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$ is *rotation-invariant*, in the following sense. A *rotation* is a linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ preserving the usual inner product $\langle x, y\rangle = \sum_i x_i y_i$, and preserving orientations (so $\det g = 1$, rather than -1). The asserted rotation-invariance is

$$\Delta(f \circ g) \;=\; (\Delta f) \circ g \qquad \text{(for twice-differentiable } f \text{ and rotation } g\text{)}$$

(In fact, Δ is also preserved by *reflections*, which are orientation-reversing, so the determinant condition can be safely ignored.)

[05.5] Check that for harmonic h and holomorphic f, the composition $h \circ f$ is invariably harmonic, while $f \circ h$ need not be. (Yes, much of the issue is suitable formulation of the computation.)

[05.6] Show that every harmonic function u on an annulus $r < |z| < R$ is of the form

$$u(z) \;=\; a_0 + b_0 \log |z| + \sum_{0 \neq n \in \mathbb{Z}} \left(a_n z^n + b_n \overline{z}^n\right)$$

for constants a_i, b_i. (*Hint:* Separate variables by writing a Fourier expansion in $\arg z$, with coefficients depending on the radial coordinate.)

[05.7] *(Euler-type equations of second order)* An ordinary differential equation of the form

$$u'' + \frac{b}{x} u' + \frac{c}{x^2} u \;=\; 0$$

with constants b, c is said to be of *Euler type*. Show that it has solutions x^α and x^β where α, β are solutions of the **auxiliary equation**

$$\lambda(\lambda - 1) + b\lambda + c \;=\; 0$$

Show that $x^\alpha \log x$ is the second solution if the root of the auxiliary equation is *double*, i.e., if $\alpha = \beta$. Use the Mean Value Theorem to genuinely prove that there are no other solutions.

[05.8] *(Rotationally invariant harmonic functions in $\mathbb{R}^n$)* For f twice-differentiable on $\mathbb{R}^n$, expressible as a (twice-differentiable) function of the *radius* r alone (at least away from 0), say f is *spherically symmetric* or *rotationally invariant*. (This could also be formulated as invariance under the action of the orthogonal group by rotations). Show that

$$\Delta f \;=\; f'' + \frac{n-1}{r} f'$$

(This is of Euler type). On $\mathbb{R}^n - \{0\}$, find two linearly independent harmonic functions.

[05.9] Prove that for non-vanishing entire f, the function $F(z) = \int_0^z \frac{f'(w)}{f(w)}\,dw$ is *entire*, and essentially gives a logarithm of f, in the sense that $f(z) = e^{C+F(z)}$ for a suitable constant C.

[05.10] Define f on the unit circle by $f(e^{i\theta}) = \theta^2$, for $-\pi < \theta < \pi$. Find a harmonic function u on the open disk whose boundary values are f.

[05.11] The Fourier expansion

$$\delta(\theta) \;=\; \sum_{n\in\mathbb{Z}} e^{in\theta} \;=\; \sum_{n\in\mathbb{Z}} \widehat{\delta}(n)\, e^{in\theta} \qquad\qquad (\text{with } \widehat{\delta}(n) = 1 \text{ for all } n \in \mathbb{Z})$$

certainly does not converge *pointwise*, but does make sense as the expansion of the periodic Dirac δ, sometimes called *Dirac comb* function on $\mathbb{R}/2\pi\mathbb{Z}$, in the following sense. The *Plancherel identity*

$$\langle u, v\rangle \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\,\overline{v(\theta)}\;d\theta \;=\; \sum_{n\in\mathbb{Z}} \widehat{u}(n)\cdot\overline{\widehat{v}(n)} \qquad\qquad (\text{for } u, v \in L^2(S^1))$$

$L^2(S^1)\times L^2(S^1) \to \mathbb{C}$ can be restricted in the first argument and extended in the second, so that for *smooth* $u(\theta) = \sum_{n\in\mathbb{Z}} \widehat{u}(n)\, e^{in\theta}$, pairing against δ correctly evaluates u at $\theta = 0$:

$$u(0) \;=\; \sum_n \widehat{u}(n)\, e^{in\cdot 0} \;=\; \sum_{n\in\mathbb{Z}} \widehat{u}(n)\cdot 1 \;=\; \sum_{n\in\mathbb{Z}} \widehat{u}(n)\cdot\overline{\widehat{\delta}(n)} \;=\; \langle u, \delta\rangle$$

Identifying the circle with the boundary $\{z : |z| = 1\}$ of the disk $\{z : |z| < 1\}$, determine the harmonic function on the disk whose boundary value function is the periodic Dirac δ.
