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Complex analysis examples discussion 01

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[This document is

http://www.math.umn.edu/~garrett/m/complex/examples_2021-22/cx_discussion_01.pdf]

[01.1] Express the two values for \sqrt{i} in terms of radicals.

The definition of square root here is that $(a+bi)^2 = i$. Multiplying out gives $(a^2 - b^2) + 2abi = i$. Separating into real and imaginary parts gives $a^2 - b^2 = 0$ and $2ab = 1$. From the first of these, $b = \pm a$, and substituting into the second gives $\pm 2a^2 = 1$. Thus,

$$\pm\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

Another approach, making more use of the geometry via the Euler identity $e^{ix} = \cos x + i \sin x$, is as follows, although a careful version of this is more verbose in this context. First, using that Euler identity, $e^{ix} = i$ exactly for $x = \frac{\pi}{2} + 2\pi n$ for integer n . Then $(e^{iy})^2 = i$ implies $e^{2iy} = e^{i(\frac{\pi}{2} + 2\pi n)}$ exactly for integers n . That is, again by Euler's identity,

$$2iy = \frac{\pi}{2} - 2\pi n \quad (\text{for integers } n)$$

and

$$\pm\sqrt{i} = e^{i(\frac{\pi}{4} + \pi n)} \quad (\text{for integers } n)$$

Since $e^{\pi in} = 1$ exactly for integers n , the complete list of *distinct* values is

$$\pm\sqrt{i} = e^{i(\frac{\pi}{4})}, e^{i(\frac{\pi}{4} + \pi)} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}, -\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

[01.2] Determine all values of 2^i , including the principal/canonical one. Determine all values of $(-1)^i$, noting that since -1 is not a positive real number, there is not principal/canonical one.

Discussion For most of this example, there is no alternative but to use Euler's identity, to raise complex numbers to the i^{th} power. That is, there is no alternative algebraic or limiting definition of α^i , unlike the case of integer, rational, or real exponents.

But, first, we do have a somewhat natural/canonical logarithm of 2, namely, the *real number* $L = \log 2$ which, exponentiated, yields 2. Then, in this narrow interpretation,

$$2^i = e^{i \cdot \log 2} = \cos L + i \sin L$$

But, honestly, in this context, there are many more values, namely, due to Euler's $e^{2\pi i} = 1$,

$$2^i = e^{i(\log 2 + 2\pi ik)} \quad (\text{for all } k \in \mathbb{Z})$$

That is,

$$2^i = e^{i(\log 2 + 2\pi ik)} = e^{i \log 2} \cdot e^{-2\pi k} \quad (\text{for all } k \in \mathbb{Z})$$

For $(-1)^i$, there is no canonical $\log(-1)$ to start from, no matter how we pretend. We can choose one, yes, easily. For example, $e^{\pi i} = -1$, so *one* of the logs of -1 is πi . Then

$$(-1)^i = e^{i(\log(-1) + 2\pi ik)} = e^{i(\pi i + 2\pi ik)} = e^{-\pi} \cdot e^{-2\pi k} \quad (\text{for all } k \in \mathbb{Z})$$

[01.3] Derive the usual formula for $\sin(z+w)$ by using e^z .

Use $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, which hold for complex z , as well, and also the opposite formula $e^{iz} = \cos z + i \sin z$. Also note that $\cos z$ is *even*, while $\sin z$ is *odd*. Then the basic property $e^{z+w} = e^z \cdot e^w$ and elementary algebra give the desired identity:

$$\begin{aligned} \sin(z+w) &= \frac{e^{i(z+w)} - e^{-i(z+w)}}{2i} = \frac{e^{iz} e^{iw} - e^{-iz} e^{-iw}}{2i} \\ &= \frac{(\cos z + i \sin z)(\cos w + i \sin w) - (\cos z - i \sin z)(\cos w - i \sin w)}{2i} \\ &= \frac{2i \sin z \cos w + 2i \cos z \sin w}{2i} = \sin z \cos w + \cos z \sin w \end{aligned}$$

[01.4] Express $\cos 5x$ as a polynomial in $\cos x$ and $\sin x$.

Again use $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $e^{iz} = \cos z + i \sin z$, and the even-ness of $\cos x$ and odd-ness of $\sin x$:

$$\begin{aligned} \cos 5x &= \frac{e^{5ix} + e^{-5ix}}{2} = \frac{(e^{ix})^5 + (e^{-ix})^5}{2} = \frac{(\cos x + i \sin x)^5 + (\cos x - i \sin x)^5}{2} \\ &= \frac{\sum_{j=0}^5 \binom{5}{j} \cos^j x (i \sin x)^{5-j} + \sum_{j=0}^5 \binom{5}{j} \cos^j x (-i \sin x)^{5-j}}{2} \\ &= \sum_{\substack{j=0 \\ \text{even}}}^5 \binom{5}{j} \cos^j x (i \sin x)^{5-j} = \cos^4 x + 10 \cos^2 x \sin^3 x + 4 \cos x \sin^4 x \end{aligned}$$

[01.5] By *mere algebra*, write a power series expansion near $z = 0$ for

$$f(z) = \frac{1}{(z-1)(z-2)}$$

This has to allow use of the geometric series expansion $\frac{1}{1-w} = 1 + w + w^2 + w^3 + \dots$ for $|w| < 1$. It is convenient to separate the two factors of the denominator by a partial fractions expansion, as

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

The coefficients can be found by looking at the asymptotics as $z \rightarrow 1$ and $z \rightarrow 2$, for example. Thus,

$$\begin{aligned} \frac{1}{(z-1)(z-2)} &= \frac{-1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} + \frac{-\frac{1}{2}}{1-\frac{z}{2}} = (1+z+z^2+\dots) - \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots) \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) z^n \end{aligned}$$

[01.6] Determine the radius of convergence of $\sum_{n \geq 1} \frac{3^n}{n(n+1)(n+2)} z^n$.

The ratio test succeeds:

$$\frac{3^n/n(n+1)(n+2)}{3^{n+1}/(n+1)(n+2)(n+3)} = \frac{1/n}{3/(n+3)} = \frac{n+3}{3n} = \frac{1}{3} + \frac{1}{n} \rightarrow \frac{1}{3} \quad (\text{as } n \rightarrow +\infty)$$

so the radius of convergence is $1/3$.

[01.7] Determine the radius of convergence of $\sum_{n \geq 1} \frac{n!}{n^n} z^n$.

The ratio test succeeds:

$$\frac{n!/n^n}{(n+1)!/(n+1)^{n+1}} = \frac{1/n^n}{(n+1)/(n+1)^{n+1}} = \left(1 + \frac{1}{n}\right)^n \rightarrow e \quad (\text{as } n \rightarrow +\infty)$$

by one characterization of the number e . That is, the radius of convergence is e .

[01.8] For two complex numbers a, b , with b not a non-positive integer, show that the radius of convergence of

$$\sum_{n \geq 0} \frac{a(a+1)(a+2) \dots (a+n-1)(a+n)}{b(b+1)(b+2) \dots (b+n-1)(b+n)} z^n$$

is at least 1.

If a happens to be a negative integer, the sum terminates, and is a polynomial, so has radius of convergence $+\infty$.

For a not a negative integer, the ratio test succeeds:

$$\begin{aligned} \frac{a(a+1)(a+2) \dots (a+n-1)(a+n)}{b(b+1)(b+2) \dots (b+n-1)(b+n)} &/ \frac{a(a+1)(a+2) \dots (a+n)(a+n+1)}{b(b+1)(b+2) \dots (b+n)(b+n+1)} \\ &= \frac{b+n+1}{a+n+1} = \frac{1 + \frac{b+1}{n}}{1 + \frac{a+1}{n}} \longrightarrow 1 \quad (\text{as } n \rightarrow +\infty) \end{aligned}$$

so the radius of convergence is 1 in this case.

[01.9] From the very definition of convergence, show that when the partial sums of a series $a_1 + a_2 + \dots$ are *bounded*, and when the elements of the sequence $\{b_n\}$ are *positive* (real) and *go to 0 monotonically*, then the series $\sum a_n b_n$ converges.

In fact, we show that the partial sums of $\sum_n a_n b_n$ form a *Cauchy* sequence, so, by the *completeness* of \mathbb{C} , the infinite sum converges. That is, we prove that the *tails* $\sum_{\ell=m}^n a_\ell b_\ell$ have the property that, given $\varepsilon > 0$, there is N such that $|\sum_{\ell=m}^n a_\ell b_\ell| < \varepsilon$ for all $m, n \geq N$.

To this end, we rearrange the tails by *partial summation*: letting $A_n = a_1 + \dots + a_n$ and $B_n = b_1 + \dots + b_n$,

$$\begin{aligned} a_m b_m + a_{m+1} b_{m+1} + \dots + a_n b_n &= (A_{m+1} - A_m) b_m + \dots + (A_{n+1} - A_n) b_n \\ &= -A_m b_m + A_{m+1} (b_m - b_{m+1}) + A_{m+2} (b_{m+1} - b_{m+2}) + \dots + A_n (b_{n-1} - b_n) + A_{n+1} b_n \end{aligned}$$

Taking absolute values, letting C be a bound for the $|A_n|$, using $b_j > b_{j+1} > 0$,

$$\begin{aligned} &\left| -A_m b_m + A_{m+1} (b_m - b_{m+1}) + \dots + A_n (b_{n-1} - b_n) + A_{n+1} b_n \right| \\ &\leq C b_m + C (b_m - b_{m+1}) + C (b_{m+1} - b_{m+2}) + \dots + C (b_{n-1} - b_n) + C b_n = C b_m \longrightarrow 0 \end{aligned}$$

since $b_m \rightarrow 0$. Thus, the partial sums are Cauchy, and the infinite sum converges.

[01.10] Show that the function $f(z) = \sum z^n/n^2$ on the open disk $|z| < 1$ extends to a *continuous* function on the *closed* unit disc.

Since the power series has radius of convergence 1, it gives an infinitely-differentiable, hence continuous, function on the *open* disk. There are several possible arguments to prove continuity on the *closed* disk, but in any case it is important to note that Abel's theorem about *non-tangential* approach does not instantly prove this, since every neighborhood of a point on the boundary circle contains points *outside* the non-tangential approach regions address.

A general argument that applies here is that a *uniform limit of continuous functions is continuous*. The finite subsums $\sum_{n \leq N} z^n/n^2$ are *polynomials*, so certainly continuous. The differences are easily bounded, with $M \leq N$, with $|z| \leq 1$, by

$$\left| \sum_{1 \leq n \leq M} \frac{z^n}{n^2} - \sum_{1 \leq n \leq N} \frac{z^n}{n^2} \right| \leq \sum_{M < n \leq N} \frac{1}{n^2}$$

Since $\sum_{n \geq 1} 1/n^2$ converges, given $\varepsilon > 0$ there is M_o such that $\sum_{M < n \leq N} \frac{1}{n^2} < \varepsilon$ for all $M_o \leq M \leq N$. Thus, the partial sums of $\sum_n z^n/n^2$ for $|z| \leq 1$ form a Cauchy sequence in sup norm, so converge to a continuous function on the closed disk.