(*September 12, 2021*)

Complex analysis examples 01

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2021-22/cx_ex_01.pdf]

If you want feedback from me on your treatment of these examples, please email me your work by Friday, Sept 17, as a PDF.

[01.1] Express the two values for $\sqrt{i}$ in terms of radicals.

[01.2] Determine all values of 2^i, including the principal/canonical one. Determine all values of $(-1)^i$, noting that since -1 is not a positive real number, there is not principal/canonical one.

[01.3] Derive the usual formula for $\cos(z+w)$ by using the exponential function.

[01.4] Express $\sin 5x$ as a polynomial in $\cos x$ and $\sin x$.

[01.5] *By mere algebra*, write a power series expansion near $z = 0$ for $f(z) = \frac{1}{(z^2-1)}$.

[01.6] Determine the radius of convergence of $\sum_{n\geq 1} \frac{1}{n(n+1)(n+2)}\, z^n$.

[01.7] Determine the radius of convergence of $\sum_{n\geq 1} \frac{n!}{n^n}\, z^n$.

[01.8] (*Hypergeometric series*) For two complex numbers a, b, with b not a non-positive integer, show that the radius of convergence of

$$\sum_{n\geq 0} \frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{b(b+1)(b+2)\ldots(b+n-1)(b+n)}\, z^n$$

is at least 1.

[01.9] (*Dirichlet's criterion/theorem*) From the very definition of convergence, show that when the partial sums of a series $a_1 + a_2 + \ldots$ are *bounded*, and when the elements of the sequence $\{b_n\}$ *are positive* (real) and *go to* 0 *monotonically*, then the series $\sum a_n b_n$ *converges*. *Hint:* summation by parts.

[01.10] Show that the function $f(z) = \sum z^n/n^2$ on the open disk $|z| < 1$ extends to a *continuous* function on the *closed* unit disc.
