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Complex analysis examples 02

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2021-22/cx_ex_02.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Oct 15.

[02.1] Parametrize counter-clockwise a circle γ of radius $r > 0$ centered at z_o , and *directly* compute $\int_{\gamma} (z - z_o)^n dz$ for all positive and negative integers n .

[02.2] Express $\sum_{n=0}^{\infty} n(n+1)z^n$ as a rational function of z .

[02.3] Express $\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)z^n$ as a rational function of z .

[02.4] Determine the Laurent expansions of $\frac{1}{1-z}$ in $|z| < 1$, and in $|z| > 1$. Observe that these two have no common region of convergence.

[02.5] Using only geometric series expansions, determine the Laurent expansion of $f(z) = 1/(z-1)(z-2)$ in the annulus $1 < |z| < 2$, and also in the annulus $|z| > 2$.

[02.6] Determine the Laurent expansion of $f(z) = 1/(z-1)^3$ in the annulus $|z| > 1$, and in the annulus $|z-1| > 0$.

[02.7] Compute the first 4 power series coefficients of $\frac{z}{e^z - 1}$ at 0. (These are essentially Bernoulli numbers. No, there is no known reduction of this computation to anything simpler.)

[02.8] Show that an entire function f satisfying $|f(z)| \leq C \cdot (1 + |z|)^r$ for some $0 \leq r \in \mathbb{R}$, and for some constant C , is a polynomial of degree at most r . (Yes, degrees of not-identically-zero polynomials are non-negative integers.)

[02.9] Compute $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$, without using arctangent.

[02.10] Compute $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$

[02.11] Compute $\int_{-\infty}^{\infty} \frac{x dx}{x^4 + 1}$ (*Hint: parity*)

[02.12] Compute $\int_{-\infty}^{\infty} \frac{e^{itx} dx}{x^2 + 1}$ with real t .

[02.13] Compute $\int_{-\infty}^{\infty} \frac{\sin(tx) dx}{x^2 + 1}$ with real t .

[02.14] Compute $\int_{-\infty}^{\infty} \frac{\cos(tx) dx}{x^2 + 1}$ with real t .

[02.15] Compute $\int_{-\infty}^{\infty} \frac{e^{itx} dx}{(x+i)^2}$ with real t .

[02.16] Compute $\int_{-\infty}^{\infty} \frac{e^{itx} dx}{(x+i)^{100}}$ with real t .

[02.17] Compute $\int_0^{\infty} \frac{x dx}{1+x^3}$

[02.18] Compute $\int_0^{\infty} \frac{x^{1/4} dx}{1+x^2}$

[02.19] Compute $\frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

[02.20] Show that $z^n + z - 1$ has n zeros inside the circle $|z| = 2$.

[02.21] Are there complex zeros of $z - \cos z$ beyond the obvious one on \mathbb{R} ?

[02.22] Let f be a continuous, bounded real-valued function on \mathbb{R} , extending to a bounded, holomorphic function on the upper half-plane \mathfrak{H} . Show f is constant.

[02.23] Evaluate the Fourier transform $\int_{-\infty}^{\infty} e^{-itx} \cdot \frac{1}{(x+i)^s} dx$ for complex s with $\operatorname{Re}(s) > 1$, using the Gamma function.
