(*November 6, 2021*)

Complex analysis examples 03

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2021-22/cx_ex_03.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Nov 12.

[03.1] Adapt the reflection principle to show that a holomorphic function on the unit disk, extending to a continuous function on the closed unit disk, with $|f(z)| = 1$ on the unit circle, extends to a holomorphic function on $\mathbb{C}$ except for finitely-many poles. (Hint: for example, $z \to \frac{z-i}{-iz+1}$ maps the real line to the unit circle.)

[03.2] Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

[03.3] Show that $\overline{\Gamma(s)} = \Gamma(\overline{s})$ for all $s \in \mathbb{C}$.

[03.4] Show that $|\Gamma(\frac{1}{2} + it)| = \sqrt{\dfrac{\pi}{\cosh \pi t}}$ for real t. (Thus, in contrast to the horizontal super-exponential *growth* of $n \to n!$, the vertical behavior is exponential decrease.)

[03.5] Prove that $f(z) = \displaystyle\int_0^1 \frac{e^{tz}\, dt}{t^2+1}$ is holomorphic on $\mathbb{C}$.

[03.6] Prove that $f(z) = \displaystyle\int_0^\infty \frac{e^{-tz}\, dt}{t^2+1}$ is holomorphic in $\mathrm{Re}(z) > 0$.

[03.7] Compute $\displaystyle\int_0^\infty \frac{x^s\, dx}{1+x^2}$

[03.8] Evaluate $\int_{-\infty}^\infty e^{2\pi i t x}\, e^{-\pi x^2}\, dx$ for real t.

[03.9] Evaluate $\int_{-\infty}^\infty e^{2\pi i t x}\, x e^{-\pi x^2}\, dx$ for real t.

[03.10] For continuous φ on the unit circle $|z| = 1$, define

$$f_\varphi(z) \;=\; \int_0^{2\pi} \frac{\varphi(e^{i\theta})}{e^{i\theta} - z}\, d\theta \qquad\qquad \text{(for } |z| < 1\text{)}$$

Show that $f(z)$ is holomorphic. Give an example of φ *not* identically 0 so that f_φ *is* identically 0.

[03.11] Show that a *real-valued* holomorphic function is constant.

[03.12] Show that a holomorphic function f with $|f(z)| = 1$, for all z, is constant.

[03.13] Show that a holomorphic function on $\mathbb{C}$ taking values in the upper half-plane is constant.

[03.14] Show that a holomorphic function taking values in the Cantor remove-middle-thirds set is constant.

[03.15] Let f be an entire function such that $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all z. Show that f is constant.