Complex analysis examples 04

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples.2021-22/cx_ex_04.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Dec 03.

[04.1] Prove that, given holomorphic \( f, g \) on a non-empty open set \( U \), and given a simple zero \( z_0 \) of \( f \), for all small-enough complex \( \varepsilon \) there is a unique zero of \( f + \varepsilon g \) nearest \( z_0 \) and it is simple.

[04.2] For small \( w \in \mathbb{C} \), let \( f(w) \) be the simple zero of \( z^5 - z + w = 0 \) near 0. Determine a few terms of the power series expansion of \( f(w) \) at \( w = 0 \).

[04.3] Exhibit a linear fractional transformation mapping 1, 2, 3 to \( z_1, z_2, z_3 \).

[04.4] Exhibit a linear fractional transformation mapping the circle \( |z| = 1 \), minus a point, to the line \( \text{Re}(z) = \text{Im}(z) \).

[04.5] Let \( z, z' \) be points in the open upper half-plane \( \mathcal{H} \). Exhibit a linear fractional transformation stabilizing \( \mathcal{H} \) and mapping \( z \) to \( z' \).

[04.6] Let \( z, z' \) be points in the open unit disk \( \mathcal{D} \). Exhibit a linear fractional transformation stabilizing \( \mathcal{D} \) and mapping \( z \) to \( z' \).

[04.7] Exhibit a conformal map of the open half disk \( \{ z : |z| < 1, \text{Re}(z) > 0 \} \) to the open unit disk.

[04.8] Exhibit a conformal map of the open unit disk with \( [0,1) \) removed to the open unit disk.

[04.9] Exhibit a conformal map of the sector \( \{ re^{i\theta} : r > 0, 0 < \theta < \frac{\pi}{4} \} \) to the open unit disk.

[04.10] Exhibit a conformal map from the strip \( \{ z = x + iy : c < ax + by < c' \} \) to the crescent

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\Omega = \{ z : |z| < 1, |z - \frac{1}{2}| > \frac{1}{2} \}
\]

[04.11] Let \( f \) be holomorphic on \( \mathbb{C} \), and meromorphic at infinity, with a pole of order \( N \). Show that \( f \) is a polynomial of degree \( N \) (and conversely).

[04.12] Let holomorphic \( f : \mathbb{CP}^1 \to \mathbb{CP}^1 \) be 2-to-1. Show that there are two linear fractional transformations \( \alpha, \beta \) such that \( \alpha \circ f \circ \beta \) is the map \( z \to z^2 \).