[06.1] Show that a harmonic function \(u\) on \(0 < |z| < 1\) such that \(\int_{|z| \leq 1} |u(z)|^2 \, dx \, dy < +\infty\) is of the form \(c \cdot \log |z| + v(z)\) where \(v\) is harmonic on \(|z| < 1\).

[06.2] Without citing Weierstraß’s or Hadamard’s product theorems, show that the infinite-product part \(\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)e^{-z/n}\) of \(1/\Gamma(z)\) is convergent for all \(z \in \mathbb{C}\).

[06.3] Show directly that \(\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) = +\infty\) and \(\prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) = 0\).

[06.4] Following Euler, show that \(\sum_{\text{prime}} \frac{1}{p}\) diverges, granting the Euler product expansion of \(\zeta(s)\) in \(\text{Re}(s) > 1\), and considering \(s \to 1^+\) along the real axis.

[06.5] From the fact that the Fourier transform of \(1/(1+x^2)\) is essentially \(e^{-|x|}\) (compute by residues), use the Poisson summation formula to evaluate \(\sum_{n=1}^{\infty} \frac{1}{1+n^2}\) in elementary terms.

[06.6] Check that the Fourier transform of the characteristic function of a symmetric interval \([-a,a]\) is \(\frac{\sin x}{x}\) (up to constants, which you should determine) (also known as sinc \(x\), up to normalizations). Express the convolution \((u * v)(x) = \int_{\mathbb{R}} u(x-y) v(y) \, dy\) of two characteristic functions \(u, v\) as an explicit piecewise-linear function, with compact support. Using \(\hat{u} \cdot \hat{v} = \hat{u} \cdot \hat{v}\) (pointwise multiplication), and Poisson summation, express \(\sum_{n} \left(\frac{\sin n}{n}\right)^2\) in elementary terms.

[06.7] For a non-vanishing holomorphic function on the open unit disk, show that \(z \to \log |f(z)|\) is a well-defined harmonic function.

[06.8] Prove that \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\) does not vanish in \(\text{Re}(s) > 1\).

[06.9] (A variant Perron identity) Show that, for \(\sigma > 0\), a vertical path integral moving upward along the line \(\text{Re}(s) = \sigma\) evaluates to

\[
\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{X^s}{s(s+\theta)} \, ds = \begin{cases} \frac{1}{\pi} (1 - X^{-\theta}) & \text{for } X > 1 \\ 0 & \text{for } 0 < X < 1 \end{cases} \quad \text{for } \theta > 0, \sigma > 0
\]

[06.10] Let \(L\) be an additive subgroup of \(\mathbb{R}\), and suppose that \(L\) is discrete as a subset of \(\mathbb{R}\), under the usual topology. Show that either \(L = \{0\}\), or \(L = \mathbb{Z} \cdot x_o\) for some \(x_o \neq 0\).
In the Gaussian integers $\mathbb{Z}[i]$, there are 4 units $\pm 1, \pm i$. The norm is $N(m + in) = m^2 + n^2$. Show that the zeta function

$$\zeta_{\mathbb{Z}[i]}(s) = \frac{1}{\#\mathbb{Z}[i]} \sum_{0 \neq m + in \in \mathbb{Z}[i]} \frac{1}{N(m + in)^s} = \frac{1}{4} \sum_{m,n \text{ not both 0}} \frac{1}{(m^2 + n^2)^s}$$

has an analytic continuation and functional equation

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Z}[i]}(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Z}[i]}(1-s)$$

by using

$$\theta(y)^2 = \left( \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} \right)^2 = \sum_{m,n \in \mathbb{Z}} e^{-\pi(m^2 + n^2)y}$$

Find a simple trick to express $\wp''$ (for a fixed lattice) as a polynomial in $\wp$.

Fix a lattice $L$. Express

$$\sum_{\lambda \in L} \frac{1}{(z - \lambda)^4} \quad \sum_{\lambda \in L} \frac{1}{(z - \lambda)^6}$$

in terms of $\wp(z)$.

Fix a lattice $\Lambda$. Show that there is no elliptic function for $\Lambda$ with exactly one pole (modulo $\Lambda$), and with that pole being simple.