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Complex analysis examples 06

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[This document is http://www.math.umn.edu/~garrett/m/complex/examples_2021-22/cx_ex_06.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Friday, Feb 25.

[06.1] Show that a harmonic function u on $0 < |z| < 1$ such that $\int_{|z|\le 1} |u(z)|^2\;dx\,dy < +\infty$ is of the form $c \cdot \log|z| + v(z)$ where v is harmonic on $|z| < 1$.

[06.2] Without citing Weierstraß's or Hadamard's product theorems, show that the infinite-product part $\prod_{n=1}^\infty \Big(1 + \dfrac{z}{n}\Big) e^{-z/n}$ of $1/\Gamma(z)$ is convergent for all $z \in \mathbb{C}$.

[06.3] Show directly that $\prod_{n=1}^\infty \Big(1 + \frac{1}{n}\Big) = +\infty$ and $\prod_{n=2}^\infty \Big(1 - \frac{1}{n}\Big) = 0$.

[06.4] Following Euler, show that $\sum_{p \text{ prime}} \frac{1}{p}$ *diverges*, granting the Euler product expansion of $\zeta(s)$ in $\mathrm{Re}(s) > 1$, and considering $s \to 1^+$ along the real axis.

[06.5] From the fact that the Fourier transform of $1/(1+x^2)$ is essentially $e^{-|x|}$ (compute by residues), use the Poisson summation formula to evaluate $\sum_{n=1}^\infty \frac{1}{1+n^2}$ in elementary terms.

[06.6] Check that the Fourier transform of the characteristic function of a symmetric interval $[-a, a]$ is $\dfrac{\sin x}{x}$ (up to constants, which you should determine) (also known as $\mathrm{sinc}\, x$, up to normalizations). Express the *convolution*

$$(u * v)(x) \;=\; \int_{\mathbb{R}} u(x-y)\, v(y)\; dy$$

of two characteristic functions u, v as an explicit piecewise-linear function, *with compact support*. Using $\widehat{u * v} = \widehat{u} \cdot \widehat{v}$ (pointwise multiplication), and Poisson summation, express $\sum_n \Big(\dfrac{\sin n}{n}\Big)^2$ in elementary terms.

[06.7] For f a *non-vanishing* holomorphic function on the open unit disk, show that $z \to \log|f(z)|$ is a well-defined harmonic function.

[06.8] Prove that $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ does not vanish in $\mathrm{Re}(s) > 1$.

[06.9] *(A variant Perron identity)* Show that, for $\sigma > 0$, a vertical path integral moving upward along the line $\mathrm{Re}(s) = \sigma$ evaluates to

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{X^s}{s(s+\theta)}\; ds \;=\; \begin{cases} \frac{1}{\theta}(1 - X^{-\theta}) & (\text{for } X > 1) \\[2ex] 0 & (\text{for } 0 < X < 1) \end{cases} \qquad (\text{for } \theta > 0,\ \sigma > 0)$$

[06.10] Let L be an additive subgroup of $\mathbb{R}$, and suppose that L is *discrete* as a subset of $\mathbb{R}$, under the usual topology. Show that either $L = \{0\}$, or $L = \mathbb{Z} \cdot x_o$ for some $x_o \neq 0$.

[06.11] In the Gaussian integers $\mathbb{Z}[i]$, there are 4 units $\pm 1, \pm i$. The *norm* is $N(m+in) = m^2+n^2$. Show that the zeta function

$$\zeta_{\mathbb{Z}(i)}(s) \;=\; \frac{1}{\#\mathbb{Z}[i]} \sum_{0\neq m+in\in\mathbb{Z}[i]} \frac{1}{N(m+in)^s} \;=\; \frac{1}{4} \sum_{m,n \text{ not both } 0} \frac{1}{(m^2+n^2)^s}$$

has an analytic continuation and functional equation

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Z}[i]}(s) \;=\; \pi^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Z}[i]}(1-s)$$

by using

$$\theta(y)^2 \;=\; \Big(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}\Big)^2 \;=\; \sum_{m,n\in\mathbb{Z}} e^{-\pi(m^2+n^2)y}$$

[06.12] Find a simple trick to express $\wp''$ (for a fixed lattice) as a polynomial in $\wp$.

[06.13] Fix a lattice L. Express

$$\sum_{\lambda\in L} \frac{1}{(z-\lambda)^4} \qquad\qquad \sum_{\lambda\in L} \frac{1}{(z-\lambda)^6}$$

in terms of $\wp(z)$.

[06.14] Fix a lattice Λ. Show that there is *no* elliptic function for Λ with exactly one pole (modulo Λ), and with that pole being *simple*.
