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Complex analysis examples 07

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[This document is http://www.math.umn.edu/˜garrett/m/complex/examples_2021-22/cx_ex_07.pdf]

If you want feedback from me on your treatment of these examples, please email your work to me by Monday, April 04.

[07.1] Let f be an entire function such that $|f(z)| \le e^{\mathrm{Re}\,(z)}$ for all z. Show that $f(z) = c \cdot e^z$ for some constant c with $|c| \le 1$. (The latter special case was on MathStackExchange, math.stackexchange.com/questions/4082085/.) More generally, suppose two entire functions f, g satisfy $|f(z)| \le |g(z)|$ for all z, and show that $f(z) = c \cdot g(z)$ for some constant c with $|c| \le 1$. (Be careful about the zeros of g.)

[07.2] Show that the group of automorphisms of the field of rational functions $\mathbb{C}(z)$ over $\mathbb{C}$ (that is, bijections $\varphi : \mathbb{C}(z) \to \mathbb{C}(z)$ which preserve addition and multiplication of rational functions, and are the identity map on the subfield $\mathbb{C}$), is the group

$$PGL_2(\mathbb{C}) \;=\; GL_2(\mathbb{C})/Z \;=\; \{\text{multiplicatively invertible complex matrices modulo the center } Z\}$$

(the center Z is the subgroup of scalar matrices) acting by linear fractional transformations $z \to \frac{az+b}{cz+d}$.

[07.3] For a fixed lattice, express $\wp(2z)$ and $\wp(3z)$ as rational functions of $\wp(z)$, using the near-algorithm that is used to prove that the field of elliptic functions for a fixed lattice is $\mathbb{C}(\wp, \wp')$. Contemplate the analogue for $\wp(nz)$.

[07.4] Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be linearly independent over $\mathbb{R}$. Show that there is a neighborhood U of $0 \in \mathbb{R}^n$ such that

$$U \cap (\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n) \;=\; \{0\}$$

[07.5] Let L be a lattice in $\mathbb{R}^n$, that is, $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$, where the v_j are linearly independent over $\mathbb{R}$. Show that

$$\sum_{0 \ne \lambda \in L} \frac{1}{|\lambda|^s} \qquad\qquad (\text{for } s \in \mathbb{C})$$

is absolutely convergent for $\mathrm{Re}\,(s) > n$. Do *not* blithely invoke some apocryphal *integral test in several variables*. Yes, for this example, the natural heuristic gives the truth, and that's a good thing. But we would *like* to prove that that heuristic gives a *proof* of the conclusion.

[07.6] From $\wp'' = 6\wp^2 - \frac{g_2}{2}$, looking at the Laurent expansion of both sides at $z = 0$, explicitly determine the rational constant C so that

$$\sum_{0 \ne \lambda \in L} \frac{1}{\lambda^8} \;=\; C \cdot \Big(\sum_{0 \ne \lambda \in L} \frac{1}{\lambda^4} \Big)^2$$

[07.7] From the previous, using Fourier expansions of Eisenstein series

$$\widetilde{E}_{2k}(z) \;=\; \sum_{c,d \text{ not both } 0} \frac{1}{(cz+d)^{2k}}$$

determine the rational constant D such that

$$\zeta(8) \;=\; D \cdot \zeta(4)^2$$

Thus, from the relatively well-known value $\zeta(4) = \pi^4/90$ (not as well known as $\zeta(2) = \pi^2/6$), we have an expression for the less-well-known $\zeta(8)$ in elementary terms. (Amazing that elliptic functions should entail relations among values of zeta!)

[07.8] For $0 \le n \in \mathbb{Z}$, let $\psi_n(z) = e^{2\pi i n z}$. In analogy with formation of Eisenstein series, the n^{th} holomorphic *Poincaré series* of weight $2k$ is (in group-theoretic form)

$$P_n(z) \;=\; \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^{2k}} \psi_n(\gamma z) \qquad \text{(with } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}\text{)}$$

and $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset \Gamma$ as usual. Since $\psi_0 = 1$, certainly $P_0 = E_{2k}$. Show that the series is absolutely and uniformly convergent for $2k \ge 4$. More interestingly, show that for $n \ge 1$ (and $2k \ge 4$) the Poincaré series is a *cuspform*. (*Hint:* Show that P_n is rapidly decreasing as $y \to +\infty$.)

[07.9] Show that $\dfrac{dx\,dy}{y^2}$ gives an $SL_2(\mathbb{R})$-invariant measure on the upper half-plane $\mathfrak{H}$. Equivalently, $\dfrac{dx \wedge dy}{y^2}$ is an $SL_2(\mathbb{R})$-invariant 2-form on $\mathfrak{H}$.

[07.10] *(Petersson inner product)* For holomorphic level-one cuspforms f, g of weight $2k$, show that

$$z \;\longrightarrow\; f(z) \cdot \overline{g(z)} \cdot y^{2k}$$

is Γ-*invariant*, that is, the action of Γ on it does *not* involve a cocycle, in contrast to the individuals f, g, y^{2k}. Show that

$$\langle f, g \rangle_{2k} \;=\; \int_{\Gamma\backslash\mathfrak{H}} f(z) \cdot \overline{g(z)} \cdot y^{2k} \frac{dx\,dy}{y^2}$$

gives a hermitian inner product on weight-$2k$ level-one cuspforms. Here the domain of integration $\Gamma\backslash\mathfrak{H}$ can be taken to be the standard fundamental domain for Γ, if desired, for concreteness.

[07.11] Show that for $n \ge 1$ and weight $2k \ge 4$, taking inner products with Poincaré series gives Fourier coefficients of cuspforms. That is, show that there is an explicit constant C_n such that, for every holomorphic weight-$2k$ level-one cuspform f,

$$\langle f, P_n \rangle_{2k} \;=\; C_n \cdot (n^{th} \text{ Fourier coefficient of } f)$$

It would be wise to use (prove?) the *unwinding*

$$\int_{\Gamma\backslash\mathfrak{H}} f(z) \cdot \overline{P_n(z)} \cdot y^{2k} \frac{dx\,dy}{y^2} \;=\; \int_{\Gamma_\infty\backslash\mathfrak{H}} f(z) \cdot \overline{\psi_n(z)} \cdot y^{2k} \frac{dx\,dy}{y^2}$$

and the fact that $\Gamma_\infty\backslash\mathfrak{H}$ has a very convenient set of representatives (*fundamental domain*)

$$F \;=\; \{z = x + iy : x \in [0,1],\; y \in (0, +\infty)\}$$

Then expand f in its Fourier expansion and integrate termwise.
