07.1 Let $f$ be an entire function such that $|f(z)| \leq e^{\text{Re}(z)}$ for all $z$. Show that $f(z) = c \cdot e^z$ for some constant $c$ with $|c| \leq 1$. (The latter special case was on MathStackExchange, math.stackexchange.com/questions/4082085/.) More generally, suppose two entire functions $f, g$ satisfy $|f(z)| \leq |g(z)|$ for all $z$, and show that $f(z) = c \cdot g(z)$ for some constant $c$ with $|c| \leq 1$. (Be careful about the zeros of $g$.)

07.2 Show that the group of automorphisms of the field of rational functions $\mathbb{C}(z)$ over $\mathbb{C}$ (that is, bijections $\varphi : \mathbb{C}(z) \to \mathbb{C}(z)$ which preserve addition and multiplication of rational functions, and are the identity map on the subfield $\mathbb{C}$), is the group

$$PGL_2(\mathbb{C}) = GL_2(\mathbb{C})/Z = \{\text{multiplicatively invertible complex matrices modulo the center } Z\}$$

(the center $Z$ is the subgroup of scalar matrices) acting by linear fractional transformations $z \to \frac{az+b}{cz+d}$.

07.3 For a fixed lattice, express $\wp(2z)$ and $\wp(3z)$ as rational functions of $\wp(z)$, using the near-algorithm that is used to prove that the field of elliptic functions for a fixed lattice is $\mathbb{C}(\wp, \wp')$. Contemplate the analogue for $\wp(nz)$.

07.4 Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be linearly independent over $\mathbb{R}$. Show that there is a neighborhood $U$ of $0 \in \mathbb{R}^n$ such that

$$U \cap (\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n) = \{0\}$$

07.5 Let $L$ be a lattice in $\mathbb{R}^n$, that is, $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$, where the $v_j$ are linearly independent over $\mathbb{R}$. Show that

$$\sum_{0 \neq \lambda \in L} \frac{1}{|\lambda|^s} \quad (\text{for } s \in \mathbb{C})$$

is absolutely convergent for $\text{Re}(s) > n$. Do not blithely invoke some apocryphal integral test in several variables. Yes, for this example, the natural heuristic gives the truth, and that’s a good thing. But we would like to prove that that heuristic gives a proof of the conclusion.

07.6 From $\wp'' = 6\wp^2 - \frac{g_2}{4}$, looking at the Laurent expansion of both sides at $z = 0$, explicitly determine the rational constant $C$ so that

$$\sum_{0 \neq \lambda \in L} \frac{1}{\lambda^s} = C \cdot \left( \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^2} \right)^2$$

07.7 From the previous, using Fourier expansions of Eisenstein series

$$\widetilde{E}_{2k}(z) = \sum_{c,d \text{ not both } 0} \frac{1}{(cz + d)^{2k}}$$
determine the rational constant \( D \) such that

\[
\zeta(8) = D \cdot \zeta(4)^2
\]

Thus, from the relatively well-known value \( \zeta(4) = \pi^4/90 \) (not as well known as \( \zeta(2) = \pi^2/6 \)), we have an expression for the less-well-known \( \zeta(8) \) in elementary terms. (Amazing that elliptic functions should entail relations among values of zeta!)

[07.8] For \( 0 \leq n \in \mathbb{Z} \), let \( \psi_n(z) = e^{2\pi inz} \). In analogy with formation of Eisenstein series, the \( n \)th holomorphic Poincaré series of weight \( 2k \) (in group-theoretic form)

\[
P_n(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^{2k}} \psi_n(\gamma z)
\]

(with \( \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \))

and \( \Gamma_\infty = \{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \} \subset \Gamma \) as usual. Since \( \psi_0 = 1 \), certainly \( P_0 = E_{2k} \). Show that the series is absolutely and uniformly convergent for \( 2k \geq 4 \). More interestingly, show that for \( n \geq 1 \) (and \( 2k \geq 4 \)) the Poincaré series is a cuspform. (Hint: Show that \( P_n \) is rapidly decreasing as \( y \to +\infty \).)

[07.9] Show that \( \frac{dx \, dy}{y^2} \) gives an \( SL_2(\mathbb{R}) \)-invariant measure on the upper half-plane \( \mathfrak{H} \). Equivalently, \( \frac{dx \wedge dy}{y^2} \) is an \( SL_2(\mathbb{R}) \)-invariant 2-form on \( \mathfrak{H} \).

[07.10] (Petersson inner product) For holomorphic level-one cuspforms \( f, g \) of weight \( 2k \), show that

\[
z \mapsto f(z) \cdot \overline{g(z)} \cdot y^{2k}
\]

is \( \Gamma \)-invariant, that is, the action of \( \Gamma \) on it does not involve a cocycle, in contrast to the individuals \( f, g, y^{2k} \). Show that

\[
\langle f, g \rangle_{2k} = \int_{\Gamma \backslash \mathfrak{H}} f(z) \cdot \overline{g(z)} \cdot y^{2k} \frac{dx \, dy}{y^2}
\]

gives a hermitian inner product on weight-\( 2k \) level-one cuspforms. Here the domain of integration \( \Gamma \backslash \mathfrak{H} \) can be taken to be the standard fundamental domain for \( \Gamma \), if desired, for concreteness.

[07.11] Show that for \( n \geq 1 \) and weight \( 2k \geq 4 \), taking inner products with Poincaré series gives Fourier coefficients of cuspforms. That is, show that there is an explicit constant \( C_n \) such that, for every holomorphic weight-\( 2k \) level-one cuspform \( f \),

\[
\langle f, P_n \rangle_{2k} = C_n \cdot (n^{th} \text{ Fourier coefficient of } f)
\]

It would be wise to use (prove?) the unwinding

\[
\int_{\Gamma \backslash \mathfrak{H}} f(z) \cdot \overline{P_n(z)} \cdot y^{2k} \frac{dx \, dy}{y^2} = \int_{\Gamma_\infty \backslash \mathfrak{H}} f(z) \cdot \overline{\psi_n(z)} \cdot y^{2k} \frac{dx \, dy}{y^2}
\]

and the fact that \( \Gamma_\infty \backslash \mathfrak{H} \) has a very convenient set of representatives (fundamental domain)

\[
F = \{ z = x + iy : x \in [0, 1], y \in (0, +\infty) \}
\]

Then expand \( f \) in its Fourier expansion and integrate termwise.