(*November 15, 2020*)

The keyhole/Hankel contour and $\zeta(-n)$

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http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/04a_keyhole_and_zeta.pdf]

The contour-integration trick illustrated here appeared in one of Riemann's proofs of analytic continuation of $\zeta(s)$. It almost immediately proves that values of $\zeta(s)$ at non-positive integers are *rational*, and shows the connection to the Laurent coefficients of $1/(e^t-1)$ at $t=0$.

[1.1] An integral representation of $\Gamma(s)\cdot\zeta(s)$ Although the integral representation of $\zeta(s)$ using a theta function is perhaps better in the long run, there is a more elementary one:

[1.1.1] Claim: For $\mathrm{Re}(s)>1$,

$$\Gamma(s)\cdot\zeta(s) \;=\; \int_0^\infty \frac{t^s}{e^t-1}\,\frac{dt}{t}$$

Proof: Expand a geometric series, exchange sum and integral, and change variables:

$$\int_0^\infty \frac{t^s}{e^t-1}\,\frac{dt}{t} \;=\; \int_0^\infty \frac{t^s\,e^{-t}}{1-e^{-t}}\,\frac{dt}{t} \;=\; \int_0^\infty t^s\Big(\sum_{n\ge 1} e^{-nt}\Big)\,\frac{dt}{t} \;=\; \sum_{n\ge 1}\int_0^\infty t^s\,e^{-nt}\,\frac{dt}{t}$$

$$=\; \sum_{n\ge 1}\frac{1}{n^s}\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t} \;=\; \Gamma(s)\cdot\sum_{n\ge 1}\frac{1}{n^s} \;=\; \Gamma(s)\cdot\zeta(s)$$

as claimed. ///

[1.2] Keyhole/Hankel contour The *keyhole* or *Hankel* contour is a path from $+\infty$ inbound along the real line to $\varepsilon>0$, counterclockwise around a circle of radius ε at 0, back to ε on the real line, and outbound back to $+\infty$ along the real line.

The usual elementary application is to evaluation of integrals similar to $\int_0^\infty \frac{t^s\,dt}{t^2+1}$, with $0<\mathrm{Re}(s)<1$. In such an example, analytically continuing counterclockwise around 0 has no impact on the denominator, but, significantly, the numerator changes by a factor $e^{2\pi i s}$, since

$$t^s \;=\; (|t|\cdot e^{i\theta})^s \;=\; |t|^s\cdot e^{i\theta s} \qquad\qquad \text{(and } \theta \text{ goes from 0 to } 2\pi)$$

We want the out-bound value of t^s to be real-valued for real s, so the inbound version of t^s must be actually be $t^s\cdot e^{2\pi i s}$. The absolute value of the integrand goes to 0 as $|t|\to 0$, so the integral over the small circle goes to 0 as $\varepsilon\to 0$, as do the integrals to and from $0,\varepsilon$ along the real line.

Thus, letting H_ε be the Hankel contour with circle of radius $\varepsilon>0$,

$$\lim_{\varepsilon\to 0}\int_{H_\varepsilon}\frac{t^s\,dt}{t^2+1} \;=\; \lim_{\varepsilon\to 0}\Big(\int_{+\infty}^{\varepsilon}\frac{(t\cdot e^{2\pi i})^s\,dt}{t^2+1} + (\text{integral over little circle}) + \int_{\varepsilon}^{+\infty}\frac{t^s\,dt}{t^2+1}\Big)$$

$$=\; (1-e^{2\pi i s})\int_0^\infty \frac{t^s\,dt}{t^2+1}$$

In this elementary example, the trick is to further modify H_ε by *not* going all the way to $+\infty$ outbound, but stopping at $+R$ for large positive R, traversing clockwise a large circle of radius R back to the positive real axis, and then inbound to ε. The integrals from R to and from $+\infty$ go to 0 as $R \to +\infty$, as does the integral over the large circle, since

$$\Big|\text{integral over big circle}\Big| \le \text{ length} \cdot \text{max value} \le \; 2\pi R \cdot \frac{R^{\mathrm{Re}\,(s)}}{R^2 - 1}$$

For each R, ε, this gives a path integral (clockwise) over a *closed* path. By *residues*, this picks up $2\pi i$ times the sum of the residues inside the path. Thus, we discover that the integrals do not depend on the parameters $0 < \varepsilon < 1 < R$. Keeping track of the relevant versions of t^s,

$$(1 - e^{2\pi i s}) \int_0^\infty \frac{t^s\,dt}{t^2+1} \;=\; 2\pi i \cdot \Big((\text{residue at } t = i) + (\text{residue at } t = -i)\Big)$$

$$=\; 2\pi i \cdot \Big(\frac{e^{\frac{3}{2}\pi i s}}{i+i} + \frac{e^{\frac{1}{2}\pi i s}}{-i-i}\Big) \;=\; \pi \cdot (e^{\frac{3}{2}\pi i s} - e^{\frac{1}{2}\pi i s})$$

That is,

$$\int_0^\infty \frac{t^s\,dt}{t^2+1} \;=\; \pi \cdot \frac{e^{\frac{1}{2}\pi i s} - e^{\frac{3}{2}\pi i s}}{1 - e^{2\pi i s}} \;=\; \pi \cdot \frac{e^{\frac{1}{2}\pi i s} - e^{-\frac{1}{2}\pi i s}}{e^{\pi i s} - e^{-\pi i s}} \;=\; \frac{\pi}{e^{\frac{1}{2}\pi i s} + e^{-\frac{1}{2}\pi i s}} \;=\; \frac{2\pi}{\cos\frac{\pi s}{2}}$$

This is a charming and useful device, but a different secondary trick is applied to $\zeta(s)$:

[1.3] Evaluation of $\zeta(-n)$

The first part of the Hankel contour discussion gives

$$\Gamma(s)\cdot\zeta(s) \;=\; \int_0^\infty \frac{t^s}{e^t-1}\,\frac{dt}{t} \;=\; \frac{1}{1-e^{2\pi i(s-1)}} \cdot \lim_{\varepsilon\to 0}\int_{H_\varepsilon} \frac{t^s}{e^t-1}\,\frac{dt}{t} \;=\; \frac{1}{1-e^{2\pi i s}} \cdot \lim_{\varepsilon\to 0}\int_{H_\varepsilon} \frac{t^s}{e^t-1}\,\frac{dt}{t}$$

For $\zeta(s)$, rewrite this as

$$\zeta(s) \;=\; \frac{1}{\Gamma(s)\cdot(1-e^{2\pi i s})} \cdot \lim_{\varepsilon\to 0}\int_{H_\varepsilon} \frac{t^s}{e^t-1}\,\frac{dt}{t}$$

At $s = -n \in \{0, -1, -2, -3, -4, \ldots\}$ two fortunate things happen. First, the pole of $\Gamma(s)$ and the zero of $1 - e^{2\pi i s}$ cancel, giving a finite, computable value. Second, the function t^{-n-1} is *single-valued*, so the inbound and outbound integrals of the Hankel contour simply *cancel* each other, *and* the integral over the small circle at 0 becomes $2\pi i$ times the residue of $\frac{t^{-n-1}}{e^t-1}$ at 0.

The periodicity of $1 - e^{2\pi i s}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi i s} \;=\; 1 - \Big(1 + \frac{2\pi i s}{1!} + \frac{(2\pi i s)^2}{2!} + \ldots\Big) \;=\; 2\pi i s + \text{higher}$$

Grant for the moment that the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\zeta(-n) \;=\; \frac{1}{\frac{(-1)^n}{n!}\cdot 2\pi i} \cdot 2\pi i \cdot \mathrm{Res}_{t=0}\frac{t^{-n-1}}{e^t-1} \;=\; (-1)^n \cdot n! \cdot \mathrm{Res}_{t=0}\frac{t^{-n-1}}{e^t-1}$$

The Laurent coefficients of $\frac{t^{-n-1}}{e^t-1}$ are more-or-less Bernoulli numbers. These are not completely elementary objects, but are certainly *rational*. Thus, $\zeta(-n) \in \mathbb{Q}$.

[1.4] Vanishing $\zeta(-2) = \zeta(-4) = \ldots = 0$

A slightly finer analysis of the generating function $\frac{1}{e^t-1}$ yields the vanishing of $\zeta(s)$ at negative even integers, as follows.

First, $\frac{1}{e^t-1}$ is very close to being *odd* as a function of t:

$$\frac{1}{e^t-1}+\frac{1}{e^{-t}-1} \;=\; \frac{1}{e^t-1}+\frac{e^t}{1-e^t} \;=\; \frac{1}{e^t-1}-\frac{e^t}{e^t-1} \;=\; \frac{1-e^t}{e^t-1} \;=\; -1$$

Thus,

$$\Big(\frac{1}{e^t-1}+\tfrac{1}{2}\Big)+\Big(\frac{1}{e^{-t}-1}+\tfrac{1}{2}\Big) \;=\; 0$$

and $\frac{1}{e^t-1}+\frac{1}{2}$ is *odd*, so all its non-vanishing Laurent coefficients are odd-degree. Thus, for even $-2n<0$,

$$\zeta(-2n) \;=\; (-1)^{2n}(2n)!\,\mathrm{Res}_{t=0}\frac{t^{-2n-1}}{e^t-1} \;=\; (2n)!\,(2n^{th}\text{ Laurent coefficient of }\frac{1}{e^t-1}) \;=\; 0$$

[1.5] Residues of $\Gamma(s)$

Finally, we determine the residues of $\Gamma(s)$. Certainly

$$\Gamma(1) \;=\; \int_0^\infty t^1\,e^{-t}\,\frac{dt}{t} \;=\; \int_0^\infty e^{-t}\,dt \;=\; 1$$

From the functional equation $s\Gamma(s)=\Gamma(s+1)$, near $s=0$

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{1+\text{higher}}{s} \;=\; \frac{1}{s}+(\text{holomorphic at } s=0)$$

Thus, the residue at 0 is 1. Iterating the functional equation,

$$\Gamma(s) \;=\; \frac{\Gamma(s+1)}{s} \;=\; \frac{\Gamma(s+2)}{(s+1)s} \;=\; \frac{\Gamma(s+3)}{(s+2)(s+1)s} \;=\; \ldots \;=\; \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+2)(s+1)s}$$

Thus, the leading Laurent term at $s=-n$ is

$$\frac{1}{s+n}\cdot\frac{\Gamma(s+n+1)}{(s+n-1)\ldots(s+2)(s+1)s}\Big|_{s=-n} \;=\; \frac{1}{s+n}\cdot\frac{\Gamma(-n+n+1)}{(-n+n-1)\ldots(-n+2)(-n+1)(-n)}$$

$$=\; \frac{1}{s+n}\cdot\frac{1}{(-1)(-2)(-3)\ldots(-n+2)(-n+1)(-n)} \;=\; \frac{1}{s+n}\cdot\frac{(-1)^n}{n!}$$

That is, the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$ as claimed.

Bibliography

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