(*October 3, 2014*)

Evaluating $\int_0^\infty \frac{\sin x}{x}\, dx$

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/04b_sinx_over_x.pdf]

The integral

$$\int_0^\infty \frac{\sin x}{x} dx$$

does not converge absolutely, but does converge. Indeed, we can similarly evaluate

$$\int_0^\infty \sin x \cdot x^s \, \frac{dx}{x} \qquad (-1 < \mathrm{Re}(s) < 1)$$

whose convergence follows by integrating by parts.

Finite integrals $\int_0^T \sin x \cdot x^s \frac{dx}{x}$ are holomorphic functions of s when $-1 < \mathrm{Re}(s)$, by checking complex differentiability directly. Since

$$\int_0^T \sin x \cdot x^s \, \frac{dx}{x} \longrightarrow \int_0^\infty \sin x \cdot x^s \, \frac{dx}{x} \qquad (\text{uniformly for } s \text{ in compacts inside } -1 < \mathrm{Re}(s) < 1)$$

the integral

$$f(s) = \int_0^\infty \sin x \cdot x^s \frac{dx}{x}$$

is a holomorphic function of s in the strip $-1 < \mathrm{Re}(s) < 1$. In particular,

$$\int_0^\infty \frac{\sin x}{x} dx = \lim_{s \to 0} \int_0^\infty x^s \, \sin x \frac{dx}{x}$$

To evaluate the integral with $\mathrm{Re}(s) < 0$, express $\sin x$ in terms of exponentials: this is legitimate for $0 < \mathrm{Re}(s)$

$$\int_0^\infty \sin x \cdot x^s \, \frac{dx}{x} = \frac{1}{2i} \left(\int_0^\infty e^{ix} \, x^s \, \frac{dx}{x} - \int_0^\infty e^{-ix} \, x^s \, \frac{dx}{x} \right) \qquad (\text{for } 0 < \mathrm{Re}(s) < 1)$$

Recall

$$\int_0^\infty e^{-Ax} \, x^s \, \frac{dx}{x} = \frac{\Gamma(s)}{A^s}$$

at first for $\mathrm{Re}(A) > 0$, by changing variables, and then for A complex with $\mathrm{Re}(A) > 0$ by the *identity principle* for analytic functions. Thus, rewrite

$$\int_0^\infty \sin x \cdot x^s \, \frac{dx}{x} = \lim_{\varepsilon \to 0^+} \frac{1}{2i} \left(\int_0^\infty e^{-(\varepsilon - i)x} \, x^s \, \frac{dx}{x} - \int_0^\infty e^{-(\varepsilon + i)x} \, x^s \, \frac{dx}{x} \right)$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{2i} \left(\frac{\Gamma(s)}{(\varepsilon - i)^s} - \frac{\Gamma(s)}{(\varepsilon + i)^s} \right) = \frac{1}{2i} \Gamma(s) \left(e^{\frac{\pi i}{2} s} - e^{-\frac{\pi i}{2} s} \right) = \Gamma(s) \cdot \sin(\frac{\pi s}{2})$$

Observe that both sides of

$$\int_0^\infty \sin x \cdot x^s \, \frac{dx}{x} = \Gamma(s) \cdot \sin(\frac{\pi s}{2}) \qquad (\text{for } 0 < \mathrm{Re}(s) < 1)$$

are holomorphic in $-1 < \mathrm{Re}(s) < 1$, since the right-hand side has a *removable* singularity at $s = 0$. The identity principle gives the equality of the two sides in the larger region $-1 < \mathrm{Re}(s) < 1$, notably including $s = 0$.

The limit $s \to 0$ picks up the pole with residue 1 of $\Gamma(s)$, and the derivative of $\sin(\pi s/2)$ at $s = 0$, so

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$
