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Asymptotics of integrals

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Watson's lemma and *Laplace's method*, the latter a simple version of *stationary phase*, are the most basic ideas in *asymptotic expansions*, after finite *Taylor-Maclaurin expansions*. [1] Watson's lemma dates from at latest [Watson 1918a], and Laplace's method at latest from [Laplace 1774]. Anachronistically, we reduce Laplace's method to Watson's lemma.

For example, a simple heuristic gives the main term [2] in the asymptotics for $\Gamma(s)$:

$$\Gamma(s) \sim \sqrt{2\pi}\, e^{-s}\, s^{s-\frac{1}{2}} \qquad \text{(as } |s| \to \infty \text{, with } \mathrm{Re}(s) \geq \delta > 0)$$

Watson's lemma gives a useful result about ratios of gamma functions, without Stirling's formula:

$$\frac{\Gamma(s+a)}{\Gamma(s)} \sim s^a \qquad \text{(as } |s| \to \infty \text{, for fixed } a \text{, for } \mathrm{Re}(s) \geq \delta > 0)$$

The specialized discussion of the Gamma function in [Whittaker-Watson 1927] or [Lebedev 1963] perhaps obscures the broader applicability of the ideas.

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[1] The simplest notion of *asymptotic* $F(s)$ for $f(s)$ as s goes to $+\infty$ on $\mathbb{R}$, or in a sector in $\mathbb{C}$, is a simpler function $F(s)$ such that $\lim_s f(s)/F(s) = 1$, written $f \sim F$. One might require an error estimate, for example,

$$f \sim F \iff f(s) = F(s) \cdot (1 + O\Big(\frac{1}{|s|}\Big))$$

A more precise form is to say that

$$f(s) \sim f_0(s) \cdot \Big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \frac{c_2}{s^{\alpha+2}} + \ldots\Big)$$

with any auxiliary function f_0, is an *asymptotic expansion* for f when

$$f = f_0(s) \cdot \Big(\frac{c_0}{s^\alpha} + \frac{c_1}{s^{\alpha+1}} + \ldots + \frac{c_n}{s^{\alpha+n}} + O\Big(\frac{1}{|s|^{\alpha+n+1}}\Big)\Big)$$

[2] The *main* term in the asymptotics for $\Gamma(s)$ is due to Stirling. Higher terms are due to Binet, and perhaps Laplace.

1. *Heuristic for the main term in asymptotics for* $\Gamma(s)$

A memorable heuristic for Stirling's formula for the main term in the asymptotics of $\Gamma(s)$, namely

$$\Gamma(s) \sim e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi} \qquad \text{(in regions } \mathrm{Re}(s) \geq \delta > 0\text{, for example)}$$

Using Euler's integral,

$$s \cdot \Gamma(s) = \Gamma(s+1) = \int_0^\infty e^{-u}\, u^{s+1}\, \frac{du}{u} = \int_0^\infty e^{-u}\, u^s\, du = \int_0^\infty e^{-u+s\log u}\, du$$

The idea is to replace the exponent $-u+s\log u$ by the quadratic polynomial in u best approximating it near its maximum, and evaluate the resulting integral. This is *justified* later in Watson's lemma and Laplace's method, but the heuristic is simpler than the justification.

More precisely, the exponent is maximum where its derivative vanishes, at the unique solution $u_o = s$ of

$$-1 + \frac{s}{u} = 0$$

The second derivative in u of the exponent is $-s/u^2$, which takes value $-1/s$ at $u_o = s$. Thus, near $u_o = s$, the quadratic Taylor-Maclaurin polynomial in t approximating the exponent is

$$-s + s\log s - \frac{1}{2!\, s} \cdot (u-s)^2$$

Thus, we *imagine* that

$$s \cdot \Gamma(s) \sim \int_0^\infty e^{-s+s\log s - \frac{1}{2s}\cdot(u-s)^2}\, du = e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-\frac{1}{2s}\cdot(u-s)^2}\, du$$

The latter integral is indeed taken over the whole real line. [3] To simplify the remaining integral, replace u by su and cancel a factor of s from both sides,

$$\Gamma(s) \sim e^{-s} \cdot s^s \cdot \int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du$$

Replacing u by $u+1$, and then u by $u \cdot \sqrt{2\pi/s}$,

$$\int_{-\infty}^\infty e^{-s(u-1)^2/2}\, du = \int_{-\infty}^\infty e^{-su^2/2}\, du = \frac{\sqrt{2\pi}}{\sqrt{s}} \int_{-\infty}^\infty e^{-\pi u^2}\, du = \frac{\sqrt{2\pi}}{\sqrt{s}}$$

In summary, the heuristic gives the correct main term of the asymptotic:

$$\Gamma(s) \sim e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

[3] Evaluation of the integral over the whole line, and simple estimates on the integral over $(-\infty, 0]$, show that the integral over $(-\infty, 0]$ is of a lower order of magnitude than the whole. Thus, the leading term of the asymptotics of the integral over the whole line is the same than the integral from 0 to $+\infty$.

2. *Watson's lemma*

The often-rediscovered *Watson's lemma*[4] gives asymptotic expansions valid in half-planes in $\mathbb{C}$ for *Laplace transform* integrals. For example, for smooth h on $(0, +\infty)$ with all derivatives of *polynomial growth*, and expressible for small $x > 0$ as

$$h(x) \;=\; x^\alpha \cdot g(x) \qquad\qquad (\text{for } x > 0, \text{ some } \alpha \in \mathbb{C})$$

where $g(x)$ is differentiable[5] on $\mathbb{R}$ near 0. Thus, $h(x)$ has an asymptotic expansion at 0^+

$$h(x) \;\sim\; x^\alpha \cdot \sum_{n=0}^{\infty} c_n\, x^n \qquad\qquad (\text{Taylor-Maclaurin } \textit{asymptotic expansion} \text{ for } x \to 0^+)$$

Watson's Lemma gives an *asymptotic expansion* of the Laplace transform of h:

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \quad\sim\quad \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \frac{\Gamma(\alpha+2)\, c_2}{s^{\alpha+2}} + \ldots \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

The error estimates below give

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-sx}\, x^\alpha\, g(x)\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha)\, g(0)}{s^\alpha} + O\Big(\frac{1}{|s|^{\mathrm{Re}(\alpha)+1}}\Big)$$

Similar conclusions hold for errors after finite sum of terms. The idea is straightforward: the expansion is obtained from

$$\int_0^\infty e^{-sx}\, h(x)\, \frac{dx}{x} \;=\; \int_0^\infty e^{-sx}\, x^\alpha \big(c_0 + \ldots + c_n x^n\big)\, \frac{dx}{x} \;+\; \int_0^\infty e^{-sx}\, x^\alpha \Big(g(x) \;-\; \big(c_0 + \ldots + c_n x^n\big)\Big)\, \frac{dx}{x}$$

The first integral gives the asymptotic expansion, and for $\mathrm{Re}(s) > 0$ the second integral can be integrated by parts and trivially bounded to give the error term. To understand the error, let $\varepsilon \geq 0$ be the smallest such that

$$N \;=\; \mathrm{Re}(\alpha) + n - \varepsilon \;\in\; \mathbb{Z}$$

The subtraction of the initial polynomial and re-allocation of the $1/x$ from the measure makes $x^{\alpha-1}(g(x) - (c_0 + \ldots + c_n x^n))$ vanish to order N at 0. This, with the exponential e^{-sx} and the presumed polynomial growth of h and its derivatives, allows integration by parts N times without boundary terms, giving

$$\int_0^\infty e^{-sx}\, h(x)\, dx \quad=\quad \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\, c_n}{s^{\alpha+n}}$$
$$+\; \frac{1}{s^N} \int_0^\infty e^{-sx} \Big(\frac{\partial}{\partial x}\Big)^N \Big(x^\alpha \cdot \big(g(x) - (c_0 + \ldots + c_n x^n)\big)\Big)\, dx$$

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[4] This lemma appeared in the treatise [Watson 1922] on page 236, citing [Watson 1918a], page 133. Curiously, the aggregate bibliography of [Watson 1922] omitted [Watson 1918a], and the footnote mentioning it gave no title. Happily, [Watson 1918a] is mentioned by title in [Bleistein-Handelsman 1975]. In the bibliography at the end, we note [Watson 1917], [Watson 1918a], [Watson 1918b].

[5] g need not be *real-analytic* near 0, only *smooth* to the right of 0, so it and its derivatives have finite Taylor-Maclaurin expansions approximating it as $x \to 0^+$.

Although the indicated leftover term is typically larger than the last term in the asymptotic expansion, it is smaller than the *next*-to-last term, so the desired conclusion holds: for $h(x)$ with asymptotic expansion at 0^+

$$h(x) \;\sim\; x^\alpha \cdot \sum_{n=0}^\infty c_n\, x^n \qquad\qquad \text{(Taylor-Maclaurin } \textit{asymptotic expansion} \text{ for } x \to 0^+)$$

and it and its derivatives of polynomial growth as $h \to +\infty$, the Laplace transform has asymptotic expansion

$$\int_0^\infty e^{-sx}\, h(x)\, dx \;=\; \frac{\Gamma(\alpha)\, c_0}{s^\alpha} + \frac{\Gamma(\alpha+1)\, c_1}{s^{\alpha+1}} + \ldots + \frac{\Gamma(\alpha+n)\, c_n}{s^{\alpha+n}} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(\alpha)+n+1}}\Big) \qquad \text{(for } n = 1,2,3,\ldots)$$

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3. *Watson's lemma and* $\Gamma(s)/\Gamma(s+a)$

A useful asymptotic awkward to derive from Stirling's formula for $\Gamma(s)$, but easy to obtain from Watson's lemma, is an asymptotic for Euler's beta integral [6]

$$B(s,a) \;=\; \int_0^1 x^{s-1}\, (1-x)^{a-1}\, dx \;=\; \frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)}$$

Fix a with $\mathrm{Re}\,(a) > 0$, and consider this integral as a function of s. Setting $x = e^{-u}$ gives an integrand fitting Watson's lemma,

$$B(s,a) \;=\; \int_0^\infty e^{-su}\, (1-e^{-u})^{a-1}\, du \;=\; \int_0^\infty e^{-su}\, (u - \frac{u^2}{2!} + \ldots)^{a-1}\, du$$

$$= \int_0^\infty e^{-su}\, u^a \cdot (1 - \frac{u}{2!} + \ldots)^{a-1}\, \frac{du}{u} \;\sim\; \frac{\Gamma(a)}{s^a} \qquad \text{(for fixed } a)$$

taking just the first term in an asymptotic expansion, using Watson's lemma. Thus,

$$\frac{\Gamma(s)\,\Gamma(a)}{\Gamma(s+a)} \;\sim\; \frac{\Gamma(a)}{s^a} \qquad \text{(for fixed } a)$$

giving

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;\sim\; \frac{1}{s^a} \qquad \text{(for fixed } a)$$

That is,

$$\frac{\Gamma(s)}{\Gamma(s+a)} \;=\; \frac{1}{s^a} + O\Big(\frac{1}{|s|^{\mathrm{Re}\,(a)+1}}\Big) \qquad \text{(for fixed } a)$$

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[6] We recall how to obtain the expression for beta in terms of gamma. With $x = u/(u+1)$ in the beta integral,

$$B(s,a) \;=\; \int_0^\infty u^{s-1}\, (u+1)^{-(s-1)-(a-1)-2}\, du \;=\; \int_0^\infty u^{s-1}\, (u+1)^{-s-a}\, du$$

$$= \frac{1}{\Gamma(s+a)} \int_0^\infty \int_0^\infty u^s\, e^{-v(u+1)}\, v^{s+a}\, \frac{dv}{v}\, \frac{du}{u}$$

using $\int_0^\infty e^{-vy}\, v^b\, dv/v = \Gamma(b)/y^b$. Replacing u by u/v gives $B(s,a) = \Gamma(s)\Gamma(a)/\Gamma(s+a)$.

4. *Main term in asymptotics by Laplace's method*

Laplace's method [7] obtains asymptotics in s for integrals

$$\int_0^\infty e^{-s\cdot f(u)}\,du \qquad\qquad (\text{for } f \text{ real-valued, } \mathrm{Re}(s)>0)$$

Information attached to u *minimizing* $f(u)$ dominate. For a *unique* minimum, at u_o, with $f''(u_o)>0$, the main term of the asymptotic expansion is

$$\int_0^\infty e^{-s\cdot f(u)}\,du \quad\sim\quad e^{-sf(u_o)}\cdot \frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \qquad\qquad (\text{for } |s|\to\infty, \text{ with } \mathrm{Re}(s)\ge\delta>0)$$

This reduces to a variant [8] of Watson's lemma, breaking the integral at points where the derivative f' changes sign, and changing variables to convert each fragment to a Watson-lemma integral. The function f must be smooth, with all derivatives of at most polynomial growth *and* at most polynomial *decay*, as $u\to+\infty$.

[4.0.1] Example: An integral $\int_0^\infty e^{-sy^2}\,h(y)\,\frac{dy}{y}$ is not quite in the form to apply the simplest version of Watson's lemma. Replacing y by $\sqrt{x}$ corrects the exponential

$$\int_0^\infty e^{-sy^2}\,h(y)\,\frac{dy}{y} \;=\; \int_0^\infty e^{-sx}\,\tfrac{1}{2}h(\sqrt{x})\,\frac{dx}{x}$$

but the asymptotic expansion of $h(\sqrt{x})$ at 0^+ will be in powers of $\sqrt{x}$. This is harmless, by a variety of possible adaptations.

[4.0.2] Remark: *In fact, in the discussion below, the odd powers of $x^{\frac{1}{2}}$ will cancel.*

For simplicity assume exactly *one* point u_o at which $f'(u_o)=0$, and that $f''(u_o)>0$, and that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$. [9] Since $f'(u)>0$ for $u>u_o$ and $f'(u)<0$ for $0<u<u_o$, there are functions F,G *smooth near* 0 such that

$$\begin{cases} F\big(\sqrt{f(u)-f(u_o)}\big) &=& u & \qquad (\text{for } u_o<u<+\infty) \\ G\big(\sqrt{f(u)-f(u_o)}\big) &=& u & \qquad (\text{for } 0<u<u_o)\end{cases}$$

Let $y=\sqrt{f(u)-f(u_o)}$ in both integrals, noting that $F(y)=u$ gives $\frac{dy}{du}\cdot F'(y)=1$, obtaining integrals *almost* as in Watson's lemma:

$$\int_0^\infty e^{-s\cdot f(u)}\,du \;=\; e^{-sf(u_o)}\Big(\int_0^{u_o} e^{-sy^2}\,du + \int_{u_o}^\infty e^{-sy^2}\,du\Big) \;=\; e^{-sf(u_o)}\int_0^\infty e^{-sy^2}\,\Big(F'(y)+G'(y)\Big)\,dy$$

$$=\; e^{-sf(u_o)}\Big(\int_0^{u_o} e^{-sy^2}\,du + \int_{u_o}^\infty e^{-sy^2}\,du\Big) \;=\; e^{-sf(u_o)}\int_0^\infty e^{-sy^2}\,y\Big(F'(y)+G'(y)\Big)\,\frac{dy}{y}$$

[7] Perhaps the first appearance of this is in [Laplace 1774].

[8] See [Miller 2006] for a thorough discussion of variants of Watson's lemma.

[9] The hypothesis of exactly *one* point u_o at which $f'(u_o)=0$, that $f''(u_o)>0$, and that $f(u)$ goes to $+\infty$ at 0^+ and at $+\infty$, holds in two important examples, namely, $f(u)=u-\log u$ for Euler's integral for $\Gamma(s)$.

Since F, G are smooth near $y = 0$, they do have Taylor-Maclaurin asymptotics in y near 0. To convert the integrals to integrals of the form in Watson's lemma, replace y by $\sqrt{x}$. This would seem to require extending Watson's lemma to tolerate asymptotic expansion of $F'(\sqrt{x}) + G'(\sqrt{x})$ in powers of $x^{\frac{1}{2}}$, but, in fact, the *odd* powers of $x^{\frac{1}{2}}$ *cancel*. Derivatives of f must increase *or decrease* only polynomially as $u \to +\infty$. An asymptotic near $x = 0$ of the form

$$\tfrac{1}{2}\big(F'(\sqrt{x}) + G'(\sqrt{x})\big) \;\sim\; c_0 + c_1 x^1 + c_2 x^2 + c_3 x^3 + \ldots \qquad (\text{as } x \to 0^+)$$

follows from a Taylor-Maclaurin expansion of $F'(y) + G'(y)$. Watson's lemma gives asymptotic expansion

$$\int_0^\infty e^{-s\cdot f(u)}\,du \;=\; e^{-sf(u_o)} \int_0^\infty e^{-sy^2}\, y\Big(F'(y)+G'(y)\Big)\,\frac{dy}{y} \;=\; e^{-sf(u_o)} \int_0^\infty e^{-sx}\, \tfrac{1}{2}x^{\frac{1}{2}}\Big(F'(\sqrt{x})+G'(\sqrt{x})\Big)\,\frac{dx}{x}$$

$$\sim\; \frac{\Gamma(\frac{1}{2})\,c_0}{s^{\frac{1}{2}}} + \frac{\Gamma(\frac{3}{2})\,c_1}{s^{\frac{3}{2}}} + \frac{\Gamma(\frac{5}{2})\,c_2}{s^{\frac{5}{2}}} + \ldots \qquad (\text{for } \mathrm{Re}(s) > 0)$$

To determine only the leading coefficient $F'(0)$, $F(y) = u$ gives $F'(y) \cdot \frac{dy}{du} = 1$, so $F'(y) = 1/\frac{dy}{du}$. From

$$y \;=\; \sqrt{f(u) - f(u_o)} \;=\; \Big(\frac{f''(u_o)}{2!}\cdot(u-u_o)^2 + O\big((u-u_o)^3\big)\Big)^{1/2}$$

$$=\; (u-u_o)\cdot\sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1 + O(u-u_o)\Big)^{1/2} \;=\; (u-u_o)\cdot\sqrt{\frac{f''(u_o)}{2}}\cdot\Big(1 + O(u-u_o)\Big)$$

the derivative is

$$\frac{dy}{du} \;=\; \sqrt{\frac{f''(u_o)}{2}} + O(u-u_o)$$

and

$$F'(y) \;=\; \frac{1}{\frac{dy}{du}} \;=\; \sqrt{\frac{2}{f''(u_o)}} \;+\; O(u-u_o)$$

At $y = 0$, also $u - u_o = 0$, so

$$F'(0) \;=\; \sqrt{\frac{2}{f''(u_o)}}$$

The same argument applied to G gives $G'(0) = F'(0)$, and Watson's lemma gives

$$\int_0^\infty e^{-s\,f(u)}\,du \;\sim\; e^{-sf(u_o)}\cdot\frac{\Gamma(\frac{1}{2})\cdot\sqrt{\frac{2}{f''(u_o)}}}{\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{f''(u_o)^{\frac{1}{2}}}\cdot\frac{1}{\sqrt{s}}$$

Last, this outcome would be obtained by replacing $f(u)$ by its quadratic approximation

$$f(u) \;\sim\; f(u_o) + \frac{f''(u_o)}{2!}\cdot(u-u_o)^2$$

Integrating over the whole line,

$$\int_{-\infty}^\infty e^{-s\cdot\big(f(u_o)+\frac{1}{2}f''(u_o)(u-u_o)^2\big)}\,du \;=\; e^{-sf(u_o)}\int_{-\infty}^\infty e^{-s\cdot\frac{1}{2}f''(u_o)(u-u_o)^2}\,du \;=$$

$$=\; e^{-sf(u_o)}\int_{-\infty}^\infty e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}f''(u_o)}}\cdot\frac{1}{\sqrt{s}} \;=\; e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}$$

This does indeed agree. Last, verify that the integral of the exponentiated quadratic approximation over $(-\infty, 0]$ is of a lower order of magnitude. Indeed, for $u \le 0$ and $u_o > 0$ we have $(u-u_o)^2 \ge u^2 + u_o^2$, and $f''(u_o) > 0$ by assumption, so

$$e^{-sf(u_o)} \int_{-\infty}^0 e^{-s\cdot\left(\frac{1}{2}f''(u_o)(u-u_o)^2\right)}\,du \;\le\; e^{-sf(u_o)} \cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2} \int_{-\infty}^0 e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du$$

$$\le\; e^{-sf(u_o)} \cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2} \int_{-\infty}^\infty e^{-s\cdot\frac{1}{2}f''(u_o)u^2}\,du \;=\; e^{-sf(u_o)} \cdot e^{-s\cdot\frac{1}{2}f''(u_o)\cdot u_o^2} \cdot \frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}} \cdot \frac{1}{\sqrt{s}}$$

Thus, the integral over $(-\infty, 0]$ has an additional exponential decay by comparison to the integral over the whole line.

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5. *Stirling's formula for main term in asymptotics for* $\Gamma(s)$

Stirling's formula for main term in asymptotics for $\Gamma(s)$ can be obtained in this context. For *real* $s > 0$, replacing u by su expresses Euler's integral for $\Gamma(s)$ as a product of an exponential and a Watson's-lemma integral:

$$s\cdot\Gamma(s) \;=\; \Gamma(s+1) \;=\; \int_0^\infty e^{-u}\,u^s\,du \;=\; \int_0^\infty e^{-u+s\log u}\,du$$

$$=\; \int_0^\infty e^{-su+s\log u+s\log s}\,s\,du \;=\; s\cdot e^{s\log s}\int_0^\infty e^{-s(u-\log u)}\,du$$

so

$$\Gamma(s) \;=\; e^{s\log s}\int_0^\infty e^{-s(u-\log u)}\,du$$

For complex s with $\mathrm{Re}(s) > 0$, both $s\cdot\Gamma(s)$ and the integral $s\cdot e^{s\log s}\int_0^\infty e^{-s(u+\log u)}\,du$ are holomorphic in s, and they *agree* for *real* s. The *identity principle* gives equality for $\mathrm{Re}(s) > 0$. With $f(u) = u - \log u$, the derivative $f'(u) = 1 - \frac{1}{u}$ has unique zero at $u_o = 1$, and $f''(1) = 0 + \frac{1}{1} = 1$. Thus,

$$\Gamma(s) \;\sim\; e^{s\log s}\cdot\left(e^{-sf(u_o)}\cdot\frac{\sqrt{2\pi}}{\sqrt{f''(u_o)}}\cdot\frac{1}{\sqrt{s}}\right) \;=\; e^{s\log s}\cdot e^{-s}\cdot\frac{\sqrt{2\pi}}{1}\cdot\frac{1}{\sqrt{s}} \;=\; \sqrt{2\pi}\,e^{(s-\frac{1}{2})\log s}\,e^{-s}$$

[5.0.1] Remark: As noted earlier, the odd powers of $x^{\frac{1}{2}}$ cancel, so $\frac{1}{2}(F'(\sqrt{x}) + G(\sqrt{x})$ has an expansion $c_0 + c_1x + c_2x^2 + \ldots$, and the error estimate in the asymptotic expansion is

$$\Gamma(s) \;\sim\; \sqrt{2\pi}\,e^{(s-\frac{1}{2})\log s}\,e^{-s}\cdot\left(1 + O\Big(\frac{1}{|s|}\Big)\right)$$

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