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Conformal mapping

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1. *Conformal (angle-preserving) maps*

A complex-valued function f on a non-empty open $U \subset \mathbb{C}$ is *conformal* if it *preserves angles*, in the sense that, for any two smooth parametrized curves $\gamma : [a, b] \to \mathbb{C}$, $\delta : [c, d] \to \mathbb{C}$ with $\gamma(a) = z_o = \delta(c)$, the angle between $\gamma'(a)$ and $\delta'(c)$ is equal to the angle between $(f \circ \gamma)'(a)$ and $(f \circ \delta)'(c)$. The function f is *orientation-preserving* if the *directed* angle is preserved.

Explicitly, for non-zero $z, w \in \mathbb{C}$,

$$\big(\text{directed angle from } z \text{ to } w\big) \quad = \quad \theta \in \mathbb{R}, \text{ such that } \frac{w}{|w|} = e^{i\theta}\frac{z}{|z|}$$

That is,

$$e^{i\theta} \;=\; \frac{w}{|w|}\Big/\frac{z}{|z|}$$

[1.0.1] Claim: A holomorphic function f is *conformal* and *orientation-preserving* at points z_o where $f'(z_o) \neq 0$.

Proof: This is a direct computation, using the chain rule, noting that f' is the *complex* derivative of f, while γ' is the *real* derivative. With $\gamma(a) = z_o = \delta(c)$,

$$(f \circ \gamma)'(a) \;=\; f'(\gamma(a)) \cdot \gamma'(a) \;=\; f'(z_o) \cdot \gamma'(a) \qquad\qquad (f \circ \delta)'(c) \;=\; f'(\delta(a)) \cdot \delta'(a) \;=\; f'(z_o) \cdot \delta'(a)$$

so

$$\frac{(f \circ \gamma)'(a)}{|(f \circ \gamma)'(a)|}\Big/\frac{(f \circ \delta)'(a)}{|(f \circ \delta)'(a)|} \;=\; \frac{f'(z_o)\,\gamma'(a)}{|f'(z_o)\,\gamma'(a)|}\Big/\frac{f'(z_o)\,\delta'(a)}{|f'(z_o)\,\delta'(a)|} \;=\; \frac{\gamma'(a)}{|\gamma'(a)|}\Big/\frac{\delta'(a)}{|\delta'(a)|}$$

showing that directed angles are preserved. ///

[1.0.2] Remark: Holomorphic f on U with non-vanishing derivative maps the mutually orthogonal families of lines $\mathrm{Re}(z) = x = \text{const}$ and $\mathrm{Im}(z) = y = \text{const}$ to mutually orthogonal curves.

For example, letting $f(z) = z^2$ on the upper half-plane, the lines $x \to x + iy_o$ become parabolas $x \to x^2 - 2ixy_o - y_o^2$ opening horizontally, and the lines $y \to x_o + iy$ become parabolas $y \to -y^2 + 2x_o y + x_o^2$ opening vertically.

Letting $f(z) = \sqrt{z}$ on $\mathbb{C} - [0, +\infty)$ amounts to looking at inverse images in the previous example, giving mutually orthogonal lines $\mathrm{Re}(z^2) = \text{const}$ and $\mathrm{Im}(z^2) = \text{const}$, namely, hyperbolas $x^2 - y^2 = \text{const}$ and hyperbolas $xy = \text{const}$.

2. *Lines and circles and linear fractional transformations*

[2.0.1] Theorem: The collection of lines and circles in $\mathbb{C} \cup \{\infty\}$ is stabilized by linear fractional transformations, and is acted upon *transitively* by them.

Proof: Clearly *affine* maps $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) = (az+b)/d$, with $ad \neq 0$, which are combinations of translations, rotations, and dilations, preserve lines and circles. Given this, a group-theoretic result greatly simplifies things:

[2.0.2] Claim: *(Bruhat decomposition)* Let P be the upper-triangular matrices in $GL_2(\mathbb{C})$, and the *Weyl element* $w_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$GL_2(\mathbb{C}) \;=\; P \sqcup P w_o P \qquad\qquad \text{(disjoint union)}$$

Proof: A group element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in P if $c = 0$, so consider $c \neq 0$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \;\in\; P$$

That is, for $g \notin P$, $gPw_o \cap P \neq \phi$. That is, $gP \cap Pw_o \neq \phi$, so $gP \cap Pw_oP \neq \phi$, and $g \in Pw_oP$. ///

Invoking the Bruhat decomposition, since $g \in P$ preserves lines and circles, it suffices to prove that the Weyl element w_o preserves lines and circles. The real and imaginary parts of $w_o(z) = 1/z$ are easily observed:

$$\frac{1}{z} \;=\; \frac{\overline{z}}{|z|^2}$$

so

$$\mathrm{Re}\,(1/z) \;=\; \frac{x}{|z|^2} \qquad\qquad \mathrm{Im}\,(1/z) \;=\; -\frac{y}{|z|^2}$$

It is convenient that w_o^2 gives the identity map, so the images and inverse images of sets under w_o are the same things. Given a line $ax + by = c$ with real a, b, c, the image of the line is

$$a\frac{x}{|z|^2} - b\frac{y}{|z|^2} \;=\; c$$

or

$$ax - by \;=\; cx^2 + cy^2$$

giving the circle

$$(x - \frac{a}{c})^2 + (y + \frac{b}{c})^2 \;=\; (\frac{a}{c})^2 + (\frac{b}{c})^2$$

The image of a circle $|z - z_o| = r$ is

$$\left|\frac{1}{z} - z_o\right| \;=\; r$$

or $|1 - z_o \cdot z| = r \cdot |z|$ or

$$1 - 2\,\mathrm{Re}\,(z_o \cdot z) + |z_o|^2 \cdot |z|^2 \;=\; r^2 \cdot |z|^2$$

In the special case that $|z_o| = r$, this is a line. Otherwise, it is a circle.

Three points on a circle determine the circle completely. A line in $\mathbb{C}$ can be viewed determined by two points on it in $\mathbb{C}$, and inevitably passing through ∞. The group of linear fractional transformation actions is *triply transitive* in the sense that it can map any triple of distinct points to any other, so is transitive on circles-and-lines. ///

[2.0.3] **Remark:** Beware: except for *affine* maps $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}(z) = (az+b)/d$, with $ad \neq 0$, linear fractional transformations do *not* respect *centers* of circles.

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3. *Elementary examples*

A *sector* is an open set of the form

$$\{z = re^{i\theta} : r > 0,\ a < \theta < b,\ \text{with } |b-a| < 2\pi\}$$

Two sectors of the special form

$$\{z = re^{i\theta} : r > 0,\ 0 < \theta < b,\ \text{with } b < 2\pi\} \qquad \{z = re^{i\theta} : r > 0,\ 0 < \theta < b',\ \text{with } b' < 2\pi\}$$

can be mapped holomorphically to each other by a *power map*

$$z \longrightarrow z^{\frac{b'}{b}} = e^{\frac{b'}{b}\cdot \log z}$$

with a continuous logarithm defined on the first sector by

$$\log z = i\pi + \int_1^{z/e^{i\pi}} \frac{dw}{w}$$

integrating along a straight line segment. In polar coordinates, this is simply

$$re^{i\theta} \longrightarrow r^{\frac{b'}{b}}\, e^{i\theta\frac{b'}{b}}$$

Exhibiting the map as a *holomorphic* map shows that it preserves angles.

Sectors with edges elsewhere than the positive real axis can be rotated, by map $z \to \mu \cdot z$ with $|\mu| = 1$, to put either edge on the positive real axis. Thus, the problem of mapping one sector to another reduces to that simpler case, by pre-composing and post-composing with rotations:

$$\{z = re^{i\theta} : r > 0,\ a < \theta < b,\ \text{with } |b-a| < 2\pi\}$$

can be mapped to another sector

$$\{z = re^{i\theta} : r > 0,\ a' < \theta < b',\ \text{with } |b'-a'| < 2\pi\}$$

by

$$\mathfrak{z} \longrightarrow \frac{z}{e^{ia}} \longrightarrow \left(\frac{z}{e^{ia}}\right)^{\frac{b'-a'}{b-a}} \longrightarrow e^{ia'} \cdot \left(\frac{z}{e^{ia}}\right)^{\frac{b'-a'}{b-a}}$$

Again, there is an unambiguous choice of continuous α^{th} power on such a sector, by

$$z^\alpha = e^{\alpha \cdot \log z}$$

where $\log z$ is defined on $\mathbb{C}$ with any ray $\{re^{i\theta_o} : r > 0\}$ removed, with this ray not lying in the given sector. Again, such a logarithm can be defined by

$$\log z \;=\; i(\theta_o + \pi) + \int_1^{z/e^{i(\theta_o+\pi)}} \frac{dw}{w}$$

integrating along a straight line segment from 1 to $z/e^i(\theta_o+\pi)$. That is, all sectors are *conformally equivalent*.

In other words, while $z \to z^\alpha$ with real $\alpha \neq 0$ is *conformal* at every point other than $z = 0$, it alters angles at 0 by multiplying angles by α.

A *bigon* is an open subset of $\mathbb{CP}^1$ bounded by two *arcs*, by which we for present purposes we mean either straight line segments, possibly infinite in one or both directions, and/or arcs of circles with angle measure $< 2\pi$.

The non-degenerate cases are specified by two vertices $z_1, z_2 \in \mathbb{PC}^1$, and two distinct circles-or-lines passing through both z_1, z_2. This configuration cuts $\mathbb{PC}$ into four connected components, each of which is a bigon.

In the degenerate case that the two points are identical, $\mathbb{CP}^1$ is cut into only three connected components.

[3.0.1] Claim: All bigons in the non-degenerate case are conformally equivalent, and are conformally equivalent to the upper half-plane.

Proof: Given vertices $z_1 \neq z_2$, map $z_1 \to 0$ and $z_2 \to \infty$ by a linear fractional transformation. Since linear fractional transformations preserve lines and circles, the boundary of the image consists of two straight lines from 0 to ∞, so the image is a *sector*. *Rotate* the sector until is is of the form $\{re^{i\theta} : r > 0,\ 0 < \theta < b,\ \text{with } b < 2\pi\}$, and the apply a *power map* to obtain the upper half-plane $\mathfrak{H}$. Thus, any non-degenerate bigon is conformally equivalent to the upper half-plane, so they are equivalent to each other. ///

[3.0.2] Claim: All degenerate bigons are conformally equivalent, and are conformally equivalent to the strip $0 < \mathrm{Re}(z) < 1$.

Proof: Map the single vertex to ∞. The two bounded arcs, mapped to arcs passing through ∞, must become straight lines with no other intersections, thus, parallel. Translate, rotate and dilate to obtain the lines $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(z) = 1$. ///

[3.0.3] Remark: An open disk can be considered as a non-degenerate bigon in many ways, as can a half-plane. *The Cayley map*

$$z \longrightarrow \frac{z+i}{iz+1}$$

maps $i \to \infty$, $-i \to 0$, and $1 \to 1$, so maps the unit circle to the real line, because linear fractional transformations preserve circles-and-lines. Thus, it maps the connected components of the complement of the circle to the connected components of the complement of the real line: since $0 \to i$, the *interior* of the circle is mapped to the *upper* half-plane.

4. $f'(z) = 0$ *implies local non-injectivity*

The following generally-useful corollary of the argument principle could have been proven earlier, but is perhaps of interest here as a sort of converse to the holomorphic inverse-function theorem:

[4.0.1] Claim: A non-constant holomorphic function f near z_o vanishing to order exactly k at z_o is $(k+1)$-to-1 in every sufficiently small punctured disk at z_o, in the following precise sense. Given a sufficiently small $r > 0$, there is a neighborhood U of z_o such that, for all z_1 in U, the value $f(z_1)$ is hit $k+1$ times inside $|z - z_o| = r$.

Proof: The *idea* is that, on one hand, since $f(z) - f(z_o)$ vanishes to order k at z_o, the argument principle would give

$$k + 1 \;\le\; \frac{1}{2\pi i}\int_\gamma \frac{(f(w) - f(z_o))'\,dw}{f(w) - f(z_o)} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f'(w)\,dw}{f(w) - f(z_o)}$$

where γ is the counterclockwise path around $|z - z_o| = r$. On the other hand,

$$z \;\longrightarrow\; \frac{1}{2\pi i}\int_\gamma \frac{f'(w)\,dw}{f(w) - f(z)}$$

should be a holomorphic function of z on some neighborhood U of z_o. An integer-valued holomorphic function is constant. Thus, it should be every value $f(z_1)$ for $z_1 \in U$ is hit at least $k+1$ times inside the circle $|z - z_o| = r$.

Two details are missing. First, we want to be sure that the denominator $f(w) - f(z_o)$ does not vanish on the circle $|z - z_o| = r$. We claim that we can shrink r if necessary so that $f(w) \neq f(z_o)$ on that circle: indeed, if there were a sequence of points $z_1, z_2, \ldots$ approaching z_o with $f(z_j) = f(z_o)$, then by the identity principle f is constant. Second, we want the denominator $f(w) - f(z)$ to not vanish for all w on the circle and for all z sufficiently near to z_o. Indeed, the set of images $I = \{f(z) : |z - z_o| = r\}$ is a continuous image of a compact set, so is compact, so is *closed*. Since $f(z_o) \notin I$, there is some neighborhood N of $f(z_o)$ disjoint from I. The inverse image $f^{-1}(N)$ is open, by continuity, and contains z_o, so there is a neighborhood U of z_o such that $f(U) \cap I = \phi$. Shrink U to be an open disk at z_o, so it is connected. ///

5. *Automorphisms of the disk and of* $\mathfrak{H}$

First, we demonstrate some explicit groups of linear transformations stabilizing the upper half-plane, or stabilizing the unit disk, in both cases large enough to act *transitively*. Then we invoke Schwarz' lemma (from the following section) to see that these groups are *all* the holomorphic automorphisms of these regions.

[5.0.1] Claim: The linear fractional transformations arising from

$$SL_2(\mathbb{R}) \;=\; \{\text{two-by-two real matrices with determinant } = 1\}$$

stabilize the upper half-plane $\mathfrak{H}$, and act transitively on it. In particular, $\mathrm{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \dfrac{\mathrm{Im}\,(z)}{|cz+d|^2}$.

Proof: First, $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$, including inverses. Directly compute the effect on imaginary parts:

$$2i\mathrm{Im}\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x+iy) \;=\; 2i\mathrm{Im}\,\frac{az+b}{cz+d} \;=\; \frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d} \;=\; \frac{(az+b)(c\overline{z}+d) - (a\overline{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{(acz\overline{z} + bc\overline{z} + adz + bd) - (acz\overline{z} + bcz + ad\overline{z} + bd)}{|cz+d|^2} \;=\; \frac{(ad-bc)(z-\overline{z})}{|cz+d|^2} \;=\; \frac{z-\overline{z}}{|cz+d|^2} \;=\; \frac{2i\mathrm{Im}\,(z)}{|cz+d|^2}$$

This shows that $SL_2(\mathbb{R})$ stabilizes the upper half-plane. To show transitivity, observe that for $x + iy \in \mathfrak{H}$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} (i) \;=\; x + iy$$

so the point $i \in \mathfrak{H}$ can be mapped to any other. ///

[5.0.2] Remark: The *special orthogonal group*

$$SO_2(\mathbb{R}) \;=\; \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R}\} \;\subset\; SL_2(\mathbb{R})$$

is the stabilizer subgroup in $SL_2(\mathbb{R})$ of the point $i \in \mathfrak{H}$.

Let $g \to g^*$ be conjugate transpose:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* \;=\; \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

and put $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[5.0.3] Claim: The subgroup

$$SU(1,1) \;=\; \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) : g^* S g = S,\; \det g = 1\} \;\subset\; GL_2(\mathbb{C})$$

stabilizes the open unit disk and acts transitively on it.

Proof: Observe that

$$-|\alpha|^2 + |\beta|^2 \;=\; \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* S \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$,

$$\big(|gz|^2 - 1\big) \cdot |cz+d|^2 \;=\; \Big(\Big|\frac{az+b}{cz+d}\Big|^2 - 1\Big) \cdot |cz+d|^2 \;=\; |az+b|^2 - |cz+d|^2$$

$$= \begin{pmatrix} az+b \\ cz+d \end{pmatrix}^* S \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \;=\; \Big(g\begin{pmatrix} z \\ 1 \end{pmatrix}\Big)^* S \Big(g\begin{pmatrix} z \\ 1 \end{pmatrix}\Big) \;=\; \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^* S g \begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} z \\ 1 \end{pmatrix}^* S \begin{pmatrix} z \\ 1 \end{pmatrix} \;=\; |z|^2 - 1 \;<\; 0 \qquad \text{(for } z \text{ in the unit disk)}$$

This proves that $U(1,1)$ stabilizes the open disk. There are rotations $z \to \mu^2 \cdot z$ given by $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$ in $SU(1,1)$ for $|\mu| = 1$, and these are transitive on each circle of radius $0 \le r < 1$. The elements

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

are in $SU(1,1)$, and send

$$0 \;\longrightarrow\; \tanh t \;=\; \frac{e^t - e^{-t}}{e^t + e^{-t}} \;=\; \frac{e^{2t} - 1}{e^{2t} + 1} \qquad \text{(for } t \in \mathbb{R}\text{)}$$

By the intermediate value theorem, since $\lim_{t\to+\infty}\tanh t = 1$, every real value T in the interval $0 \le T < 1$. Thus, $SU(1,1)$ is transitive on the open unit disk. ///

[5.0.4] Remark: Conjugation by the Cayley map $C = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ has the property that $C \cdot g \cdot C^{-1} \in SL_2(\mathbb{R})$ for $g \in SU(1,1)$, and *vice-versa*. In particular, C^{-1} conjugates rotations of the disk to the group $SO_2(\mathbb{R})$.

[5.0.5] Corollary: *(of Schwarz' lemma)* Any holomorphic automorphism of the open unit disk is given by an element of $SU(1,1)$. Any holomorphic automorphism of the upper half-plane is given by an element of $SL_2(\mathbb{R})$.

Proof: Given a holomorphic map f of the open disk to itself, compose with an element of $U(1,1)$ to adjust so that $f(0) = 0$. Certainly $|f(z)| \le 1$ for $|z| < 1$. By Schwarz' lemma, $|f(z)| \le |z|$ and $|f'(0)| \le 1$, and if equality holds at any point with $0 < |z| < 1$ or if $|f'(0)| = 1$, then $f(z) = \mu \cdot z$ with $|\mu| = 1$.

By the holomorphic inverse function theorem, $(f^{-1})'(0) = 1/f'(0)$. Also, Schwarz' lemma applies to f^{-1}. Thus,

$$1 \;\le\; \frac{1}{|f'(0)|} \;=\; |(f^{-1})'(0)| \;\le\; 1$$

implies equality, and that $f(z) = \mu \cdot z$ for some $|\mu| = 1$. This shows that the linear fractional transformations given by $SU(1,1)$ are the whole holomorphic automorphism of the open unit disk.

Similarly, for a holomorphic automorphism f of $\mathfrak{H}$, let $f(i) = z$, and let $g \in SL_2(\mathbb{C})$ map $g(z) = i$. Then

$$h \;=\; C^{-1} \circ g \circ f \circ C$$

is a holomorphic automorphism of the open unit disk fixing 0, so is a rotation coming from $SU(1,1)$. Note that $CgC^{-1} \in SO_2(\mathbb{R})$. Then

$$f \;=\; g \circ (C \circ h \circ C^{-1})$$

expresses $f : \mathfrak{H} \to \mathfrak{H}$ as a composition of linear fractional transformations $g \in SL_2(\mathbb{R})$. ///

[5.0.6] Remark: The action on $\mathbb{CP}^1$ given by *scalar* elements of $GL_2(\mathbb{C})$ is *trivial*, so the central subgroups of $U(1,1)$ and of $SL_2(\mathbb{R})$ act trivially.

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6. *Schwarz' lemma*

[6.0.1] Theorem: For f holomorphic on $|z| < 1$, with bound $|f(z)| \le 1$, and with $f(0) = 0$,

$$|f(z)| \;\le\; |z| \qquad \text{and} \qquad |f'(0)| \;\le\; 1$$

Equality $|f(z_o)| = |z_o|$ holds for some $0 < |z_o| < 1$ if and only if $f(z) = \mu \cdot z$ for some constant μ with $|\mu| = 1$.

Proof: The function $F(z) = f(z)/z$ has a removable singularity at $z = 0$, and takes value $f'(0)$ there. For each $0 < r < 1$,

$$|F(z)| \;=\; \frac{|f(z)|}{r} \;\le\; \frac{1}{r} \qquad \text{(on the circle } |z| = r\text{)}$$

By the maximum modulus principle, $|F(z)| \le 1/r$ on $|z| \le r$. Thus, for all r' with $0 < r < r' < 1$, $|F(z)| \le 1/r'$ on $|z| \le r$, so $|F(z)| \le 1$ on $|z| \le r$. This holds for every $0 < r < 1$, so $|F(z)| \le 1$ on $|z| < 1$. This already gives $f'(0) \le 1$.

If $|F(z_o)| = 1$ for some $0 < |z_o| < 1$, or if $|f'(0)| = 1$, then $|F|$ attains its sup in the interior, so is a constant μ, and $f(z) = \mu \cdot z$. ///
