(*February 12, 2015*)

Poisson summation

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/09b_PoissonSummation.pdf]

The simplest *Poisson summation formula* is

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n) \qquad\qquad (\text{suitable } f, \text{ Fourier transform } \widehat{f})$$

with Fourier transform $f \to \widehat{f}$ normalized as

$$\text{Fourier transform of } f \;=\; \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i x\xi}\, dx$$

The simplest heuristic for the truth of Poisson summation is in terms of representability of functions by Fourier *series*, as follows. Given f a function on $\mathbb{R}$, *periodicize* f

$$F(x) \;=\; \sum_{n\in\mathbb{Z}} f(x+n)$$

The periodic function F should be represented by its *Fourier series*, so

$$F(x) \;=\; \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x} \int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx$$

The Fourier *coefficients* of F expand to be seen as the Fourier *transform* of f:

$$\int_0^1 F(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n)\, e^{-2\pi i\ell x}\, dx \;=\; \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \int_{\mathbb{R}} f(x)\, e^{-2\pi i\ell x}\, dx \;=\; \widehat{f}(\ell)$$

Evaluating at 0 should give

$$\sum_{n\in\mathbb{Z}} f(n) \;=\; F(0) \;=\; \sum_{\ell\in\mathbb{Z}} \widehat{f}(\ell)$$

Under what hypotheses is this legitimate? Certainly f must be of sufficient decay so that the integral for its Fourier transform is (absolutely?) convergent, and so that summing its translates by $\mathbb{Z}$ is nicely convergent. We'd want f to be continuous, probably differentiable, so that we can talk about pointwise values of F, and so that the Fourier series of F converges pointwise to F.

For f and several derivatives rapidly decreasing, the Fourier transform $\widehat{f}$ will be of sufficient decay so that its sum over $\mathbb{Z}$ does converge. In this vein, a simple sufficient hypothesis for convergence is that f be in the *Schwartz space* of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, that is,

$$\text{Schwartz space} \;=\; \{\text{smooth } f \;:\; \sup_x\, (1+x^2)^\ell |f^{(i)}(x)| < \infty \text{ for all } i, \ell\}$$

This assumption is too restrictive for some applications, and, indeed, is needlessly restrictive, but is simple and robust, and allows straightforward proofs.

The least obvious ingredient is pointwise convergence of Fourier series. Recall a simple elementary version:

[0.0.1] Claim: For periodic f piecewise-C^o functions left-continuous and right-continuous at its discontinuities, for points x_o at which f is C^0 and *left-differentiable* and *right-differentiable*, the Fourier series of f evaluated at x_o *converges* to $f(x_o)$:

$$f(x_o) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{2\pi i n x_o}$$

That is, for such functions, at such points, the Fourier series *represents* the function *pointwise.* ///

We want to be sure *not only* that the periodicization F of a Schwartz function f is $C^1(S^1)$, but that the sum expressing it *converges to it* in a C^1-topology, to be able to differentiate termwise. The finite partial sums $\sum_{|n|\le N} f(x+n)$ are *not periodic*, so cannot converge to F in $C^1(S^1)$, since they're not in that space. Nevertheless, we can show that the partial sums converge in the *Fréchet* space $C^1(\mathbb{R})$, whose topology is given by *semi-norms*

$$\nu_T(f) \;=\; \sup_{|x|\le T} |f(x)| + \sup_{|x|\le N} |f'(x)| \qquad\qquad (\text{for } T=1,2,\ldots)$$

The topology cannot be given by a single *norm*, so is not *Banach*, but it is complete-metrizable, with many different metrics, no *canonical* one, all approximately of the form

$$d(f,g) \;=\; \sum_{N\ge 1} 2^{-N}\, \frac{\nu_N(f-g)}{1+\nu_N(f-g)}$$

The periodicity of the *full* sum is separate from the issue of convergence in $C^1(\mathbb{R})$.

The rapid decay certainly implies, for example, that there are constants A, B such that

$$|f(x)| \;\le\; A(1+|x|)^{-2} \qquad\qquad \text{and} \qquad\qquad |f'(x)| \;\le\; B(1+|x|)^{-2}$$

Thus, the tails are estimated by

$$\sup_{|x|\le T} \Big| \sum_{M<|n|\le N} |f(x+n)| + |f'(x+n)| \Big| \;\le\; (A+B) \sup_{|x|\le T} \sum_{M<|n|\le N} (1+|x|+|n|)^{-2}$$

$$\le\; (A+B) \sum_{M<|n|\le N} (1+|n|)^{-2} \;\le\; (A+B)\int_M^\infty \frac{dt}{t^2} \;\le\; (A+B)\frac{1}{M}$$

This can be made arbitrarily small by taking M large, giving convergence with respect to the T^{th} seminorm, hence, with in the metrizable topology on $C^1(\mathbb{R})$. ///

A similar argument shows that the periodicization is smooth.
