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Infinitude of zeros in the critical strip

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Hadamard's theorem on canonical products yields a short proof that $\zeta(s)$ has infinitely-many zeros in the critical strip $0 \le (\mathrm{Re}\,(s) \le 1$. This is essentially an echo of [Titchmarsh 1986], page 30, and some background.

Hadamard's product theorem, for growth order $\lambda \in \mathbb{R}$, asserts that for the integer h satisfying $h \le \lambda < h+1$, an entire function f of order λ has product expansion

$$f(z) \;=\; e^{g(z)} \cdot z^\nu \prod_{z_i} \Big(1 - \frac{z}{z_i}\Big)\, e^{p_h(z/z_i)}$$

where ν is the order of 0 at 0, z_i runs through non-zero zeros of f, $g(z)$ is a polynomial of degree at most h, and $p_h(z)$ is the h^{th} truncation of the Taylor series for $log(1-z)$, namely,

$$p_h(z) \;=\; z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^h}{h}$$

For $h = 0$, take $p_0(z) = 0$. Hadamard's theorem controls the leading exponential: rather than being $e^{g(z)}$ with some unfathomable entire function $g(z)$, we have sharp constraints on $g(z)$.

Thus, there is the peculiar corollary that entire functions of growth order $\lambda < 1$ have $h = 0$, so have very simple product expansions

$$f(z) \;=\; e^a \cdot z^\nu \prod_{z_i} \Big(1 - \frac{z}{z_i}\Big) \qquad\qquad \text{(for } f \text{ entire of order } \lambda < 1)$$

for some constant a. In particular, if f is not a *polynomial*, then it has infinitely-many zeros.

This corollary can be used to prove that $\zeta(s)$ has infinitely-many zeros in the strip $0 \le \mathrm{Re}\,(s) \le 1$, as follows.

From the functional equation, and from the fact that $\Gamma(s)$ has no zeros, the only possible zeros of $\xi(s)$ are in $0 \le \mathrm{Re}\,(s) \le 1$.

Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. In light of the functional equation $\xi(1-s) = \xi(s)$ and the fact that $\xi(s)$ has exactly two poles, at $s = 0, 1$, which are simple, the function $s(1-s)\xi(s)$ is *entire* and still satisfies the same equation. That is $z \to (\frac{1}{2}+z)(\frac{1}{2}-z)\xi(\frac{1}{2}+z)$ is entire and *even*. Thus, it is a function of z^2, and there is an entire function f such that

$$f(z^2) \;=\; (\tfrac{1}{2}+z)(\tfrac{1}{2}-z)\xi(\tfrac{1}{2}+z)$$

There is a traditionally-defined function $\Xi(z)$ which differs from this f only in normalization. We have shown that $\xi(s)$ is of growth-order 1, so f is of growth-order $\frac{1}{2}$. Thus, by the corollary to Hadamard's theorem, either f is a polynomial, or has infinitely-many zeros. If $f(z)$ were a polynomial, then $f(z^2)$ would be a polynomial, as well. But the super-polynomial growth of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ for s real and $s \to +\infty$ shows that this is impossible. Thus, f has infinitely-many zeros. ///

[Titchmarsh 1986] E. C. Titchmarsh, edited and with a preface by D. R. Heath-Brown, The theory of the Riemann zeta- function, 2nd edn The Clarendon Press/Oxford University Press, New York, 1986. First edition, 1951, successor to *The Zeta-Function of Riemann*, 1930.