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Fundamental domains for $SL_2(\mathbb{Z})$ and Γ_θ

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1. *Fundamental domain for $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$*

The simplest beginning choice of discrete subgroup Γ of $G = SL_2(\mathbb{R})$ is

$$\Gamma \;=\; SL_2(\mathbb{Z}) = \{\text{2-by-2 integer matrices with determinant 1}\}$$

Both for use below and to show that $SL_2(\mathbb{Z})$ is a large group, note:

[1.0.1] Claim: Given *relatively prime* integers c, d, there are integers a, b such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Proof: For *any* integers c, d, there are integers m, n such that

$$\text{greatest common divisor } c, d \;=\; m \cdot c + n \cdot d$$

Here the greatest common divisor is 1, so take $a = n$, $b = -m$, and then $ad - bc = 1$. ///

To be able to draw a picture of the quotient, we take an archaic approach which nevertheless succeeds in this case, namely, we find a *fundamental domain* for Γ on $\mathfrak{H}$, meaning to find a *nice* set of representatives for the quotient. Second, see how the edges of the fundamental domain are glued together when mapped to the quotient $\Gamma\backslash\mathfrak{H}$.

[1.0.2] Claim: Every Γ-orbit in $\mathfrak{H}$ has a representative in

$$\overline{F} \;=\; \{z \in \mathfrak{H} : |z| \geq 1,\; |\mathrm{Re}(z)| \leq \frac{1}{2}\}$$

More precisely, each orbit has a *unique* representative in the *standard fundamental domain*

$$F \;=\; \{z \in \mathfrak{H} : |z| > 1,\; -\frac{1}{2} \leq \mathrm{Re}(z) < \frac{1}{2}\} \cup \{z \in \mathfrak{H} : |z| = 1, \mathrm{Re}(z) \leq 0\}$$

Proof: From above, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\mathrm{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$$

The set of complex numbers $cz + d$ is a subset of the lattice $\mathbb{Z} \cdot z + \mathbb{Z} \subset \mathbb{C}$. Since it is a discrete *subgroup*, it has (at least one) smallest (in absolute value) non-zero element.

Thus, $\inf |cz + d| = \min |cz + d| > 0$, taking the infimum or minimum over *relatively prime* c, d, which we have observed are exactly the lower rows of elements of Γ. Then

$$\sup \frac{1}{|cz+d|} = \max \frac{1}{|cz+d|} < \infty$$

Thus, for fixed $z \in \mathfrak{H}$,

$$\sup \mathrm{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) \;=\; \sup \frac{\mathrm{Im}\, z}{|cz+d|^2} = \max \frac{\mathrm{Im}\, z}{|cz+d|^2} < \infty$$

Thus, in each Γ-orbit there is (at least one) point z assuming the maximum value of $\mathrm{Im}\, z$ on that orbit.

Since $\mathrm{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \mathrm{Im}\, z/|cz+d|^2$, for z giving maximal $\mathrm{Im}\, z$ in its orbit, it must be that

$$|cz+d| \;\geq\; 1$$

for all c, d relatively prime. Thus, for example, for $d = 0$ there is the *inversion*

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(z) \;=\; -1/z$$

Thus, $|1 \cdot z + 0| \geq 1$, so for $\mathrm{Im}\, z$ maximal in its Γ-orbit, $|z| \geq 1$.

We can adjust any $z \in \mathfrak{H}$ by

$$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}(z) \;=\; z + n \qquad (\text{for } n \in \mathbb{Z})$$

to normalize $-1/2 \leq \mathrm{Re}(z) < 1/2$.

So take $|z| \geq 1$ and $|\mathrm{Re}(z)| \leq 1/2$ and show that $|cz+d| \geq 1$ for *all* c, d. Break z into its real and imaginary parts $z = x + iy$. Then

$$|cz+d|^2 \;=\; (cx+d)^2 + c^2y^2 \;=\; c^2(x^2+y^2) + 2cdx + d^2 \;\geq\; c^2(x^2+y^2) - |cd| + d^2$$

$$\geq\; c^2(|z|^2 - \frac{1}{4}) + \frac{c^2}{4} - |cd| + d^2 \;\geq\; c^2(|z|^2 - \frac{1}{4})$$

Thus, for $|c| \geq 2$, we have $|cz+d| > 1$ when $|z| \geq 1$ and $|x| \leq 1/2$.

For $c = 0$, necessarily $d = \pm 1$, and the only corresponding elements of Γ are

$$\begin{bmatrix} \pm 1 & n \\ 0 & \pm 1 \end{bmatrix}$$

The only z's with $|z| \geq 1$ and $|x| \leq 1/2$ that can be mapped to each other by such group elements are $-\frac{1}{2} + iy$ and $\frac{1}{2} + iy$. We whimsically keep the former as our chosen representative.

For $c = \pm 1$,

$$|cz+d|^2 \;=\; 2xd + d^2 + |z|^2 \geq -|d| + d^2 + 1 \;\geq\; 1 \qquad\qquad (\text{for } d \in \mathbb{Z})$$

In fact, for $|x| < 1/2$, there is a *strict* inequality

$$2xd + d^2 + |z|^2 \;>\; -|d| + d^2 + 1 \;\geq\; 1$$

so $|cz+d| > 1$. When $|x| = 1/2$, still $-|d| + d^2 + 1 > 1$, *except* for $d = 0, \pm 1$.

Thus, first without worrying about strictness of the inequalities, $|cz+d| \geq 1$ for $|z| \geq 1$ and $|x| \leq 1/2$, and the set $\overline{F}$ contains (at least one) representative for every orbit. What remains is to eliminate duplicates.

We have already observed that the only duplicates for $|z| > 1$ have $|x| = 1/2$, and $z \to z+1$ maps the $x = -1/2$ line to the $x = 1/2$ line.

Now consider $|z| = 1$. For $|x| < 1/2$, the only cases where $|cz + d| = 1$ are with $c = \pm 1$ and $d = 0$, which correspondes to matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} * & \pm 1 \\ \mp 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix} \qquad \text{(for some } n \in \mathbb{Z})$$

For $|z| = 1$, the inversion $z \to -1/z$ maps $z = x + iy$ to

$$-\frac{1}{z} = -\overline{z}/|z|^2 = -\overline{z} = -x + iy$$

Thus, for $|x| < 1/2$, the only one among these products that maps z back to the fundamental domain is exactly the inversion $z \to -1/z$. This inversion identifies the two arcs

$$\{|z| = 1 \ \text{ and } \ -\frac{1}{2} \le x \le 0\} \qquad \{|z| = 1 \ \text{ and } \ 0 \le x \le \frac{1}{2}\}$$

Thus, we should include only one or the other of these two arcs in the strict fundamental domain.

Last, with $|z| = 1$ and $|x| = 1/2$, there are exactly four group elements modulo $\pm 1_2$ (the center $\{\pm 1_2\}$ acts trivially) that map z to the closure of the fundamental region. These are: the identity, one of the translations $z \to z \pm 1$, the inversion $z \to -1/z$, and the *composite* of the translation and the inversion. That is, in addition to the identity,

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{map } -\frac{1}{2} + \frac{i\sqrt{3}}{2} \ \text{ to the boundary of } \overline{F}$$

and

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{map } \frac{1}{2} + \frac{i\sqrt{3}}{2} \ \text{ to the boundary of } \overline{F}$$

Thus, in the quotient $\Gamma \backslash \mathfrak{H}$, the identification of the sides $x = \pm 1$ creates a (topological) cyclinder, and the identification of the two arcs on the bottom closes the bottom of the cylinder. Thus, topologically, we have a cylinder closed at one end, which is a disk. But the non-euclidean geometry (if we were to pay more attention to details) suggests that the *top* of the cylinder is infinitely far away, and the radius of the cylinder goes to 0 as one goes toward the open top end, so it is more accurate to think of the quotient $\Gamma \backslash \mathfrak{H}$ as a raindrop shape. ///

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2. *Inversion and translation generate* $SL_2(\mathbb{Z})$

[2.0.1] Claim: *The inversion (long Weyl element) $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and translations $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with $n \in \mathbb{Z}$ generate $\Gamma = SL_2(\mathbb{Z})$.*

Proof: Again use the fact that $\mathbb{Z} \cdot z + \mathbb{Z}$ is a *lattice* in $\mathbb{C}$. In particular, there is *no* infinite sequence of decreasing sizes $|c_1 z + d_1| > |c_2 z + d_2| > \ldots$ with integers c_j, d_j. Thus, there is no infinite *increasing* sequence of heights

$$\frac{y}{|c_1 z + d_1|^2} < \frac{y}{|c_2 z + d_2|^2} < \ldots$$

Since $\mathrm{Im}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z)\right) = \dfrac{y}{|cz+d|^2}$, this implies that there is *no* infinite increasing sequence

$$\mathrm{Im}(\gamma_1 z) < \mathrm{Im}(\gamma_2 z) < \ldots \qquad (\text{for } \gamma_j \in \Gamma)$$

This promises that the following procedure does eventually put every point $z \in \mathfrak{H}$ inside the standard fundamental domain for Γ.

Given $z \in \mathfrak{H}$, translate z to z_1 satisfying $|\mathrm{Re}(z_1)| \le \frac{1}{2}$. If $|z_1| \ge 1$, stop: z_1 is in the fundamental domain. If $|z_1| < 1$, apply the inversion, noting

$$\mathrm{Im}\Big(\frac{-1}{z_1}\Big) \;=\; \frac{\mathrm{Im}(z_1)}{|z_1|^2} \;>\; \mathrm{Im}(z_1) \qquad\qquad (\text{since } |z_1| < 1)$$

Continue: translate $-1/z_1$ back to z_2 in the strip. If $|z_2| \ge 1$, stop. If $|z_2| < 1$, invert. Translate back to z_3 in the strip, and so on. The sequence $\mathrm{Im}(z_1) < \mathrm{Im}(z_2) < \ldots$ must be *finite*, so the process terminates after finitely many steps.

Thus, given $\gamma \in \Gamma$, take z in the *interior* of the fundamental domain, and let δ be a finite product of inversions and integer translations so that $\delta^{-1}\gamma z$ is back in the fundamental domain. Since z is in the interior, $\delta^{-1}\gamma = \pm 1_2$. Since $w^2 = -1_2$, necessarily γ is expressible in terms of inversions and integer translations. ///

[2.0.2] **Remark:** The number of steps require to move a given $z \in \mathfrak{H}$ into the fundamental domain is not simple to describe. This complication is visible in pictures of the tiling of the upper half-plane by images of the fundamental domain.

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3. *Fundamental domain for* Γ_θ *and* $\Gamma(2)$

The determination of the standard fundamental domain F for $\Gamma(1) = SL_2(\mathbb{Z})$ allows explicit determination of fundamental domains for finite-index *subgroups* such as the *principal congruence subgroups*

$$\Gamma(N) \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$$

by choosing coset representatives γ_i for $\Gamma(N)$ in $\Gamma(1)$, and then [1]

$$\text{fundamental domain for } \Gamma(N) \;=\; \bigcup_i \gamma_i F$$

It is useful that $\Gamma(N)$ is exactly the kernel of the group homomorphism

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N) \qquad\quad \text{by} \qquad\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \bmod N & b \bmod N \\ c \bmod N & d \bmod N \end{pmatrix}$$

so is *normal* in $\Gamma(1)$.

For the important special choice [2]

$$\Gamma_\theta \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2\}$$

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[1] Since $\mathfrak{H} = \bigcup_{\gamma\in\Gamma(1)} \gamma\overline{F}$, for representatives γ_i with $\Gamma(1) = \bigcup_i \Gamma(N)\gamma_i$,

$$\mathfrak{H} \;=\; \bigcup_{\gamma\in\Gamma(1)} \gamma\overline{F} \;=\; \bigcup_{\gamma\in\bigcup_i \Gamma(N)\gamma_i} \gamma\overline{F} \;=\; \bigcup_{\gamma\in\Gamma(N)} \bigcup_i \gamma\gamma_i\overline{F} \;=\; \bigcup_{\gamma\in\Gamma(N)} \gamma\Big(\bigcup_i \gamma_i\overline{F}\Big)$$

[2] This subgroup Γ_θ is important because it appears in sums-of-squares problems, the simplest application of *theta series* to seemingly elementary number-theory problems.

$$= \quad \Gamma(2) \quad \cup \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \Gamma(2)$$

the coset-representative oriented choice of fundamental domain can be adjusted to prove the corollary that Γ_θ is generated by $z \to -1/z$ and $z \to z+2$, as below.

[3.0.1] Remark: The following assertion holds without assuming p is prime, but all we need at the moment is $p = 2$, in any case. Further, the surjectivity of $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/2)$ is easy to observe directly, since, for example, the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

surject to $SL_2(\mathbb{Z}/2)$.

[3.0.2] Claim: For p prime, the natural map

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/p) \quad \text{is surjective}$$

Proof: Let q be the quotient map $\mathbb{Z} \to \mathbb{Z}/p$. First, given u, v not both 0 in $\mathbb{Z}/p$, we will find *relatively prime* c, d in $SL_2(\mathbb{Z})$ such that $qc = u$ and $qd = v$.

For $v \notin p\mathbb{Z}$, there is $0 \neq d \in R$ such that $qd = v$. Consider the conditions on $c \in R$

$$c = u \bmod p \qquad \text{and} \qquad c = 1 \bmod d$$

As $d \notin p\mathbb{Z}$, by the maximality of the ideal $p\mathbb{Z}$ there are $x \in \mathbb{Z}$ and $pm \in p\mathbb{Z}$ such that $xd + pm = 1$. Let $c = xdu + pm$. From $xd + pm = 1$, $xd = 1 \bmod pm$ and $pm = 1 \bmod d$, so this expression for c satisfies the two congruences conditions. In particular, $qc = u$, and since $c = 1 \bmod d$ it must be that $\gcd(c, d) = 1$.

For $v = 0$ in $\mathbb{Z}/p$, necessarily $u \neq 0$, and we reverse the roles of c, d in the previous paragraph.

Thus, there are coprime c, d in $\mathbb{Z}$ whose images mod p are u, v. For integers s, t there exist a, b such that $\gcd(s, t) = as - bt$. The coprimality of c, d implies that there are a, b in R such that $ad - bc = 1$. That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ u & v \end{pmatrix} \mod p$$

Further adjustment to accommodate the *upper* row is more straightforward: Given $\begin{pmatrix} r & s \\ u & v \end{pmatrix}$ in $SL_2(\mathbb{Z}/p)$, and letting $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ also denote its image in $SL_2(\mathbb{Z}/p)$,

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} v & -b \\ -u & a \end{pmatrix} = \begin{pmatrix} rv - su & * \\ uv - vu & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

The right-hand side is in $SL_2(\mathbb{Z}/p)$, so, in fact, it must be of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod p$$

So

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \mod p$$

giving the surjectivity. ///

[3.0.3] Claim: $\#SL_2(\mathbb{Z}/p) = (p^2-1)p$ for prime p.

Proof: First, count $GL_2(\mathbb{Z}/p)$. This is the number of ordered bases for the vector space $(\mathbb{Z}/p)^2$ over $\mathbb{Z}/p$, since an element of $GL_2(\mathbb{Z}/p)$ sends one basis to another, is transitive on ordered bases, and $g \in GL_2(\mathbb{Z}/p)$ fixes a basis v_1, v_2 only for $g = 1_2$.

The first basis element v_1 can be any non-zero vector in $(\mathbb{Z}/p)^2$, giving p^2-1 choices. For each such choice, the second basis element can be anything not on the $\mathbb{Z}/p$-line spanned by v_1, giving p^2-p choices. Thus, $\#GL_2(\mathbb{Z}/p) = (p^2-1)(p^2-p)$.

The determinant map surjects $GL_2(\mathbb{Z}/p) \to (\mathbb{Z}/p)^\times$, and has kernel $SL_2(\mathbb{Z})$, so the *index* of $SL_2(\mathbb{Z}/p)$ is $\#(\mathbb{Z}/p)^\times = p-1$, and the cardinality is as claimed. ///

[3.0.4] Corollary: $\Gamma(2)$ has six coset representatives in $\Gamma(1)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Proof: The index is $(2^2-1)2 = 6$. The six listed matrices are in $SL_2(\mathbb{Z})$ and are distinct mod 2. ///

[3.0.5] Corollary: Γ_θ has three coset representatives in $\Gamma(1)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof: The index is 3, since Γ_θ is index 2 above $\Gamma(2)$. The three listed matrices are in $SL_2(\mathbb{Z})$ and are not only distinct mod 2 but also do not differ mod $\Gamma(2)$ merely by multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. ///

[3.0.6] Corollary: A fundamental domain for Γ_θ is

$$F_\theta \;=\; \{z \in \mathfrak{H} \;:\; |z| \ge 1 \text{ and } |\mathrm{Re}(z)| \le 1\}$$

Proof: With standard fundamental domain

$$F \;=\; \{z \in \mathfrak{H} \;:\; |z| \ge 1 \text{ and } |\mathrm{Re}(z) \le \tfrac{1}{2}\}$$

for $\Gamma(1)$, the coset representatives for Γ_θ in $\Gamma(1)$ give a fundamental domain

$$F' \;=\; F \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F \cup \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} F$$

for Γ_θ. *[... iou ...]* pictures! We will symmetrize this into a more easily-describable form. With hindsight, we replace

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The point is that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F$ is understandable as a translate of the inverted F.

Move the *right half* of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F \cup \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} F$ left by $z \to z - 2$, so that the two halves are symmetric about the imaginary axis. This produces the region claimed in the theorem. ///

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4. *Generators for* Γ_θ

[4.0.1] Corollary: Inversion $z \to -1/z$ and translation $z \to z + 2$ generate Γ_θ.

Proof: Given $z \in \mathfrak{H}$, translate z by $2\mathbb{Z}$ to $|\mathrm{Re}(z)| \le 1$. If $|z| \ge 1$, stop. If not, invert, and then translate back to $|\mathrm{Re}(z)| \le 1$. This produces a sequence of points $z_1, z_2, \ldots$ with

$$\mathrm{Im}(z_1) \;<\; \mathrm{Im}(z_2) \;<\; \ldots$$

As earlier, $\mathrm{Im}(z_n)$ is of the form $\mathrm{Im}(z)/|cz + d|^2$, and any such sequence must be finite. That is, inversion and translation by $1\mathbb{Z}$ eventually put z into the fundamental domain for Γ_θ.

Given $\gamma \in \Gamma_\theta$, choose z in the interior of the fundamental region, and let δ be a composition of inversions and translations by $2\mathbb{Z}$ so that $\delta^{-1}\gamma z$ is back in the fundamental domain. Then $\delta^{-1}\gamma = \pm 1_2$, so $\gamma = \pm\delta$. Since the inversion squares to -1_2, $\gamma \in \Gamma_\theta$. ///
