(*March 24, 2015*)

Level-one elliptic modular forms

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[This document is
http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/10d_level_one.pdf]

In this very brief introduction, we omit discussion of *Hecke operators*, *Poincaré series*, and *theta series*.

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1. *Automorphy condition, Fourier expansion, cuspforms*

An *elliptic (holomorphic) modular form* of *level one* and *weight* $2k$ is a *holomorphic* function f on the upper half-plane $\mathfrak{H}$ meeting the *automorphy condition*

$$f(\gamma z) \;=\; (cz+d)^{2k}\cdot f(z) \qquad\qquad (\text{for } z\in\mathfrak{H} \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}))$$

with $\gamma z = \frac{az+b}{cz+d}$, and meeting the *growth condition* that it is *bounded* on the closure of the standard fundamental domain

$$F \;=\; \{z\in\mathfrak{H} \;:\; |z|>1,\; |\mathrm{Re}(z)|<\tfrac{1}{2}\}$$

The function

$$j \;:\; SL_2(\mathbb{Z})\times\mathfrak{H} \;\longrightarrow\; \mathbb{C}^\times \qquad \text{by} \qquad j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \longrightarrow cz+d$$

is the *cocycle*. When context makes the details clear, the modifier *elliptic* is often dropped. [1]

$$f|_{2k}\gamma \;\;=\;\; f(\gamma z)\cdot(cz+d)^{-2k} \qquad\qquad (\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

for arbitrary complex-valued functions f on $\mathfrak{H}$, allowing the automorphy condition to be rewritten as

$$f|_{2k}\gamma \;\;=\;\; f \qquad\qquad (\text{for all } \gamma\in SL_2(\mathbb{Z}))$$

[1.0.1] Note: The holomorphic modular forms of weight $2k$ for $SL_2(\mathbb{Z})$ form a complex vector space under value-wise *sums*. Also, the *product* of a weight $2k$ form and a weight $2k'$ form gives a weight $2k+2k'$ form.

[1.0.2] Remark: The modifier *elliptic modular* refers to the fact that the function is on $\mathfrak{H}$, as opposed to some other homogeneous space, and is *holomorphic*, as opposed to meeting some other local analytic condition. *Level one* refers to the automorphy requirement for all $\gamma\in SL_2(\mathbb{Z})$ rather than some smaller or different subgroup of $SL_2(\mathbb{R})$.

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[1] Traditional terminology is that $f\to f|_{2k}\gamma$ is the *slash* operator, although this name fails to suggest any meaning other than reference to the notation itself. In fact, obviously $f(z)\to f(\gamma z)(cz+d)^{-2k}$ is a *left translation* operator, albeit complicated by the automorphy factor. That is, this is a *right* action of $SL_2(\mathbb{Z})$ on functions on $\mathfrak{H}$, while the group action of $SL_2(\mathbb{Z})$ on $\mathfrak{H}$ is written on the *left*.

[1.0.3] **Remark:** Boundedness in the closure of the fundamental domain does *not* imply boundedness on $\mathfrak{H}$, because modular forms are not quite *invariant* under $SL_2(\mathbb{Z})$, but only almost invariant, with the cocycle making things more complicated.

[1.1] Fourier expansions

The upper-triangular element $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ sends $z \to z+1$, and $j(\gamma, z) = 1$, so a level-one modular form f has the property

$$f(z+1) \;=\; f(\gamma z) \;=\; j(\gamma,z)^{2k} \cdot f(z) \;=\; 1^{2k} \cdot f(z) \;=\; f(z)$$

That is, modular forms are *periodic* in $x = \mathrm{Re}(z)$, with period 1. Thus, as functions of z, modular forms have *Fourier expansions* in x, with coefficients depending on $y = \mathrm{Im}(z)$:

$$f(x+iy) \;=\; \sum_{n\in\mathbb{Z}} c_n(y)\, e^{2\pi i n x}$$

Since f is *holomorphic*, it satisfies the Cauchy-Riemann equation

$$\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) f(x+iy) \;=\; 0$$

Differentiating term-wise,

$$0 \;=\; \sum_{n\in\mathbb{Z}} \Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\Big(c_n(y) e^{2\pi i n x}\Big) \;=\; \sum_{n\in\mathbb{Z}} \Big(2\pi i n c_n(y) e^{2\pi i n x} + i c_n'(y) e^{2\pi i n x}\Big)$$

By uniqueness of Fourier expansions,

$$2\pi i n c_n(y) + i c_n'(y) \;=\; 0 \qquad\qquad (\text{for all } n \in \mathbb{Z})$$

This is a linear, constant-coefficient differential equation for $c_n(y)$:

$$c_n'(y) + 2\pi n c_n(y) \;=\; 0$$

Thus,

$$c_n(y) \;=\; \text{constant } \times e^{-2\pi n y}$$

and the Fourier expansion of a (holomorphic) modular form is of the form

$$f(z) \;=\; \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n z} \qquad\qquad (\text{constants } c_n \in \mathbb{C})$$

[1.1.1] **Remark:** Fourier expansions of modular forms are sometimes called *q-expansions*, with $q = e^{2\pi i z}$.

[1.2] Fourier expansions and growth condition

Use the standard notation

$$A_n \;\ll\; B_n$$

for the assertion that $|A_n| \le C \cdot B_n$ for some constant C.

[1.2.1] **Proposition:** A modular form $f(z) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n z}$ has $c_n = 0$ for $n < 0$, and $|c_n| \ll e^{2\pi n}$ for $n \ge 0$, with implied constant depending on f.

Proof: Let $|f(z)| \leq C$ for z in the fundamental domain. Then the usual expression for the n^{th} Fourier component gives

$$|c_n| e^{-2\pi ny} \;=\; \Big| \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi inx} f(x+iy)\,dx \Big| \;\leq\; \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| e^{-2\pi inx} f(x+iy) \Big|\,dx \;\leq\; \int_{-\frac{1}{2}}^{\frac{1}{2}} 1\cdot C\,dx \;\leq\; C$$

That is,

$$|c_n| \;\leq\; e^{2\pi ny}\cdot C$$

As $y \to +\infty$ with $z \in F$, we find $c_n = 0$ for $n < 0$. For $n \geq 0$, taking $y = 1$ gives the estimate. ///

[1.2.2] Remark: The estimate $|c_n| \ll e^{2\pi n}$ is very bad, but useful in preliminaries.

[1.3] Cuspforms

A modular form with 0^{th} Fourier coefficient 0 is a *cuspform.*

This innocent cuspform condition, beyond holomorphy, automorphy, and the growth condition, has important ramifications later.

[1.3.1] Theorem: *(Hecke)* A weight $2k$ holomorphic cuspform f has *exponential decay*

$$|f(x+iy)| \;\ll_f\; e^{-2\pi y} \qquad\qquad (\text{as } y \to +\infty)$$

with implied constant depending on f. The Fourier coefficients c_n of f satisfy

$$|c_n| \;\ll\; n^k$$

Proof: Using the preliminary bound $|c_n| \ll e^{2\pi ny}$ from above,

$$|f(z)| \;\ll\; \sum_{n\geq 1} e^{2\pi n}\, e^{-2\pi ny} \;=\; \sum_{n\geq 1} e^{-2\pi n(y-1)} \;=\; \frac{e^{-2\pi(y-1)}}{1-e^{-2\pi(y-1)}}$$

by summing the geometric series, giving the exponential decay. Since

$$|\mathrm{Im}\,(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z)| \;=\; \frac{\mathrm{Im}\,(z)}{|cz+d|^2}$$

the function $y^k \cdot |f(z)|$ is $SL_2(\mathbb{Z})$-*invariant*, rather than merely satisfying the automorphy condition. Due to the exponential decay in the fundamental domain, $y^k \cdot |f(z)|$ is surely *bounded* in the fundamental domain. By $SL_2(\mathbb{Z})$-invariance, $y^k \cdot |f(z)|$ is *bounded* on $\mathfrak{H}$.

For any $y > 0$,

$$|c_n \cdot e^{-2\pi ny}| \;\leq\; \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big| e^{-2\pi inx}\, f(x+iy) \Big|\,dx \;\ll_f\; y^{-k}$$

That is, $|c_n| \ll_f y^{-k} e^{2\pi ny}$. The bounding expression blows up as $y \to 0^+$ and as $y \to +\infty$, but we can find its minimum: solve

$$0 \;=\; \frac{\partial}{\partial y}\Big(y^{-k} e^{2\pi ny}\Big) \;=\; -ky^{-k-1} e^{2\pi ny} + 2\pi n y^{-k} e^{2\pi ny} \;=\; (-k + 2\pi ny) y^{-k-1} e^{2\pi ny}$$

The minimizing $y = k/2\pi n$ gives

$$|c_n| \;\ll\; \Big(\frac{k}{2\pi n}\Big)^{-k} e^{2\pi n\cdot \frac{k}{2\pi n}} \;=\; n^k \cdot \Big(\frac{2\pi e}{k}\Big)^k$$

giving the asserted bound. ///

[1.3.2] Remark: [Hecke 1937]'s bound given above was improved by [Rankin 1939] and [Selberg 1940]. [Ramanujan 1916]'s and [Petersson 1930]'s conjecture that $|c_p| \leq 2p^{k-\frac{1}{2}}$ for prime p and weight $2k$ cuspforms, was proven by [Deligne 1974] as application of his completion of proof of the *Weil conjectures.*

2. *Explicit example: holomorphic Eisenstein series*

One normalization of *(holomorphic) Eisenstein series* is

$$E_{2k}(z) \;=\; \tfrac{1}{2} \sum_{\text{coprime } c,d} \frac{1}{(cz+d)^{2k}}$$

Legitimate analogues of an integral test show that this is absolutely convergent, and uniformly so for z in compacts, for $2k \ge 4$. Thus, E_{2k} is holomorphic. [2]

[2.0.1] Remark: Unless $2k$ is an integer, there are serious problems with the definition of the $2k^{th}$ powers. When $2k \ge 3$ is an odd integer, the pairs (c,d) and $(-c,-d)$ produce terms that cancel each other, and the expression is identically 0.

As earlier, direct computation shows that

$$E_{2k}(\gamma z) \;=\; (cz+d)^{2k} E_{2k}(z) \qquad\qquad (\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

Namely, with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$E_{2k}(\gamma z) \;=\; \tfrac{1}{2} \sum_{\text{coprime } C,D} \frac{1}{(C\frac{az+b}{cz+d} + D)^{2k}} \;=\; (cz+d)^{2k} \sum_{\text{coprime } C,D} \frac{1}{(C(az+b) + D(cz+d))^{2k}}$$

$$= \; (cz+d)^{2k} \sum_{\text{coprime } C,D} \frac{1}{((Ca+Dc)z + (Cb+Dd))^{2k}}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ Ca+Dc & Cb+Dd \end{pmatrix}$$

Thus, the map $(C,D) \to (Ca+Dc, Cb+Dd)$ is a bijection on the set of coprime integers, and we have $(cz+d)^{2k} E_{2k}(z)$. [3]

The leading fraction and the coprimality condition are elementary shadows of a more meaningful expression,

$$E_{2k}(z) \;=\; \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(cz+d)^{2k}} \qquad\qquad (\text{with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

[2] An infinite sum $\sum_{n \ge 1} f_n$ of holomorphic functions, if uniformly absolutely convergent on compacts, is again holomorphic. This follows from Morera's theorem, that a function f is holomorphic if its integrals over small triangles are 0. Namely, any given triangular path γ traces out a compact set, so, given $\varepsilon > 0$, there is N such that $\sum_{n \ge N} |f_n(z)| < \varepsilon$ for all z on γ, and the integral of this *tail* over γ is at most ε times the length of γ. Since the finite sum $\sum_{n \le N} f_n$ is holomorphic, its integral over γ is 0. Thus, the integral over every triangle is smaller than every positive real, so is 0.

[3] The same computation demonstrates the *cocycle relation* $j(gh,z) = j(g,hz)j(h,z)$ for $g,h \in SL_2(\mathbb{R})$ and $z \in \mathfrak{H}$. This certifies that the action $f \to f|_{2k}\gamma$ has the *associativity*

$$(f|_{2k}\gamma)|_{2k}\delta \;=\; f|_{2k}(\gamma\delta)$$

necessary for this to be a legitimate *right* action.

where $\Gamma = SL_2(\mathbb{Z})$, $\Gamma_\infty = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma\}$. Indeed, for integers c, d to be the lower row of an element $\gamma \in \Gamma$, necessarily c, d are coprime. With even integer $2k$, changing c, d for $-c, -d$ does not change $(cz+d)^{2k}$. And, given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$ in Γ,

$$\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} * & * \\ cd-dc & * \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \Gamma_\infty$$

proving the bijection.

So $E_{2k}(z)$ satisfies the automorphy condition.

Thus, $E_{2k}(z)$ meets the holomorphy condition and the automorphy condition. Demonstration that it is bounded in the closure of the standard fundamental domain would complete proof that it is an elliptic modular form.

This demonstration is postponed till after computation of the Fourier coefficients of holomorphic Eisenstein series below.

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3. *Divisor/dimension formula, applications*

A useful relation on the *orders of vanishing* of an elliptic modular form f of weight $2k$ for $SL_2(\mathbb{Z})$ is produced via the *argument principle*, by path-integration of $f'(z)/f(z)$ around the boundary of a height-T truncation

$$F_T \;=\; \{|z| \geq 1,\ |\mathrm{Re}(z)| \leq \tfrac{1}{2},\ \mathrm{Im}(z) \leq T\}$$

of the standard fundamental domain F.

The *divisor* of a function is the set of it zeros, counting order-of-vanishing, that is, counting multiplicities. [4] Less usually, the *order of vanishing* at $i\infty$, $\nu_f(i\infty)$, of $f(z) = \sum_n c_n e^{2\pi i n z}$ is the smallest n_o such that $c_n = 0$ for $n < n_o$. Still, this is consistent with the usual notion by viewing the Fourier expansion as a power series in $q = e^{2\pi i z}$.

[3.0.1] Theorem: Let $\nu_f(z)$ be the order of vanishing of not-identically-zero f at $z \in \mathfrak{H}$. Including only an *irredundant* collection of representatives for $SL_2(\mathbb{Z})\backslash\mathfrak{H}$,

$$\frac{\nu_f(i)}{2} \;+\; \frac{\nu_f(\rho)}{3} \;+\; \nu_f(i\infty) \;+\; \sum_{\text{other } z} \nu_f(z) \;=\; \frac{2k}{12}$$

where ρ is a cube root of unity in $\mathfrak{H}$ and f is weight $2k$. *(Proof in following section.)*

This divisor relation yields important corollaries.

[3.1] The first cuspform A small further preparation: Ramanujan's $\Delta(z)$-function is a non-zero constant multiple of $E_4^3 - E_6^2$, which the proof of the following shows to be not identically zero. The choice of the

[4] As usual in complex analysis, at a point $z_o \in \mathfrak{H}$, the *order of vanishing* $\nu_f(z_o)$ of a holomorphic function f is the smallest n_o so that the n_o^{th} power series coefficient of f at z_o is non-zero. That is, with

$$f(z) \;=\; \sum_{n=0}^{\infty} c_n\,(z-z_o)^n$$

the *order* (of vanishing) of f at z_o is the smallest n_o such that $c_{n_o} \neq 0$.

multiplying constant is usually to make $\Delta(z)$ have Fourier expansion (with vanishing 0^{th} Fourier coefficient, and) 1^{st} Fourier coefficient 1:

$$\Delta(z) \;=\; 1\cdot e^{2\pi i z} + \sum_{n\ge 2} \tau(n)\, e^{2\pi i n z}$$

The higher Fourier coefficients are sometimes denoted $\tau(n)$ for reasons of tradition. When we compute the Fourier coefficients of E_{2k}, we will see that they are of the form

$$E_{2k}(z) \;=\; 1\cdot e^{2\pi i\cdot 0\cdot z} + \sum_{n\ge 1} c_n\, e^{2\pi i n z}$$

Granting this, since there are no negative-index Fourier components,

$$E_4(z)^3 - E_6(z)^2 \;=\; (1+\text{higher})^3 - (1+\text{higher})^2 \;=\; (1+\text{higher}) - (1+\text{higher})$$

$$=\; \text{vanishing } 0^{th} \text{ Fourier component} + \text{higher Fourier components}$$

Thus, granting this feature of the Fourier expension of Eisenstein series, the constant multiple $\Delta(z)$ of $E_4(z)^3 - E_6(z)^2$ is indeed a *cuspform.*

[3.1.1] Corollary: The spaces M_{2k} of modular forms of weight $2k$ for $SL_2(\mathbb{Z})$ are $\{0\}$ for $2k<0$ or $2k$ an *odd* integer. In small non-negative weights: $M_0 = \mathbb{C}$ and $M_2 = \{0\}$, while for even integer weights $2k \ge 4$,

$$M_{2k} \;=\; \mathbb{C}\cdot E_{2k} \;\oplus\; \Delta\cdot M_{2k-12}$$

That is, for weights up through 22,

$$\begin{cases} M_0 &=& \mathbb{C} \\ M_2 &=& \{0\} \\ M_4 &=& \mathbb{C}\cdot E_4 \\ M_6 &=& \mathbb{C}\cdot E_6 \\ M_8 &=& \mathbb{C}\cdot E_8 \\ M_{10} &=& \mathbb{C}\cdot E_{10} \\ M_{12} &=& \mathbb{C}\cdot E_{12} \;\oplus\; \mathbb{C}\cdot\Delta \\ M_{14} &=& \mathbb{C}\cdot E_{14} \\ M_{16} &=& \mathbb{C}\cdot E_{16} \;\oplus\; \mathbb{C}\cdot\Delta E_4 \\ M_{18} &=& \mathbb{C}\cdot E_{18} \;\oplus\; \mathbb{C}\cdot\Delta E_6 \\ M_{20} &=& \mathbb{C}\cdot E_{20} \;\oplus\; \mathbb{C}\cdot\Delta E_8 \\ M_{22} &=& \mathbb{C}\cdot E_{22} \;\oplus\; \mathbb{C}\cdot\Delta E_{10} \end{cases}$$

Proof: For *odd* integers $2k$ (momentarily resisting the suggestion of the notation that it's an even integer), and $f\in M_{2k}$,

$$f(z) \;=\; f\Big(\frac{-z+0}{0\cdot z-1}\Big) \;=\; f\Big(\begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix} z\Big) \;=\; (0\cdot z-1)^{2k}\cdot f(z) \;=\; (-1)\cdot f(z)$$

so $f(z)=0$.

For even integer $2k$, the point is that, for small non-negative even integers $2k$, it is not easy to meet the condition

$$\frac{n_i}{2} + \frac{n_\rho}{3} + n_{i\infty} + \sum_{\text{other } z} n_z \;=\; \frac{2k}{12}$$

with non-negative integers n_*.

To begin the more serious discussion, for $2k = 0$, all orders of vanishing must be 0, since they are non-negative integers. *Constants* are obviously in M_0. The trick is that, for a holomorphic modular form f of weight 0, $f(z) - f(z_o)$ vanishes at z_o for every $z_o \in \mathfrak{H}$. Thus, $f(z)$ is identically equal to $f(z_o)$, that is, is constant.

For $2k = 2$, there is *no* collection of orders of vanishing combining to give the required $2k/12 = 1/6$, so $M_2 = \{0\}$.

For $2k = 4$, on one hand, the only way to get $4/12 = 1/3$ is

$$\underbrace{\frac{0}{2}}_{\text{at } i} + \underbrace{\frac{1}{3}}_{\text{at } \rho} + \underbrace{0}_{\text{at } i\infty} + \sum_{\text{other } z} 0 \;=\; \frac{4}{12}$$

On the other hand, we are granting ourselves that the holomorphic Eisenstein series E_4 is in M_4, so evidently $E_4(\rho) = 0$, and the vanishing is just first-order. Given $f \in M_4$, take $z_o \in \mathfrak{H}$ not in the Γ-orbit of ρ, and consider

$$f_2 \;=\; f \;-\; \frac{f(z_o)}{E_4(z_o)} \cdot E_4$$

By design, f_2 vanishes at z_o:

$$f_2(z_o) \;=\; f(z_o) \;-\; \frac{f(z_o)}{E_4(z_o)} \cdot E_4(z_o) \;=\; 0$$

Such vanishing can occur only for f_2 identically zero, so f is a constant multiple of E_4.

Similarly, for $2k = 6, 8, 10$, there is only one way to satisfy the divisor relation:

$$\begin{cases} \underbrace{\frac{1}{2}}_{\text{at } i} + \underbrace{\frac{0}{3}}_{\text{at } \rho} + \underbrace{0}_{\text{at } i\infty} + \sum_{\text{other } z} 0 \;=\; \frac{6}{12} \\[2ex] \underbrace{\frac{0}{2}}_{\text{at } i} + \underbrace{\frac{2}{3}}_{\text{at } \rho} + \underbrace{0}_{\text{at } i\infty} + \sum_{\text{other } z} 0 \;=\; \frac{8}{12} \\[2ex] \underbrace{\frac{1}{2}}_{\text{at } i} + \underbrace{\frac{2}{3}}_{\text{at } \rho} + \underbrace{0}_{\text{at } i\infty} + \sum_{\text{other } z} 0 \;=\; \frac{10}{12} \end{cases}$$

and $E_{2k} \in M_{2k}$. The same argument as for M_4 shows that every element of M_6, M_8, M_{10} is a constant multiple of E_6, E_8, E_{10}.

Things change at M_{12}, since $12/12 = 1$: there is no numerical obstacle to vanishing at $i\infty$ and other points, in addition to the special points i and ρ. Still, E_{12} is present, and we are granting in advance that its Fourier expansion is of the form

$$E_{12}(z) \;=\; 1 \cdot e^{2\pi i \cdot 0 \cdot z} \;+\; \sum_{n \geq 1} a_n \, e^{2\pi i n z}$$

Given $f \in M_{12}$ with Fourier expansion

$$f(z) \;=\; \sum_{n \geq 0} b_n \, e^{2\pi i n z}$$

substract a multiple of E_{12} to make the 0^{th} Fourier coefficient 0: consider

$$f_2(z) \;=\; f(z) \;-\; b_0 \cdot E_{12}$$

Thus, $\nu_{f_2}(i\infty) = 1$, and f_2 is a *cuspform*, by definition. The divisor relation shows that f_2 has no *other* zeros, *unless* by mischance f_2 is identically 0.

To prove *existence* of a not-identically-zero cuspform of weight 12, note that $E_4^3 - E_6^2$ is weight 12, and has 0^{th} Fourier coefficient 0, so is a candidate. To show that $E_4^3 - E_6^2$ is not identically 0, recall from above that $E_4(\rho) = 0$ and does not vanish otherwise, while $E_6(i) = 0$ and does not vanish otherwise. Thus, $E_4^3 - E_6^2$ cannot vanish at either ρ or i, so is not identically 0. Up to normalizing constant, $\Delta = E_4^3 - E_6^2$.

By the divisor relation, Δ *only* vanishes at $i\infty$, and there to order 1. Now we will see that $M_{12} = \mathbb{C}E_{12} + \mathbb{C}\Delta$. Given $f \in M_{12}$, as before, subtract a multiple E_{12} to make the 0^{th} Fourier coefficient of $f_2 = f - cE_{12}$ be 0. Then *divide* f_2 by Δ, taking advantage of the fact that Δ does not vanish in $\mathfrak{H}$, and vanishes only to first order at $i\infty$. Thus, f_2/Δ is in $M_0 = \mathbb{C}$, proving that f_2 is a multiple of Δ, and $M_{12} = \mathbb{C}E_{12} + \mathbb{C}\Delta$.

Similarly, now that the non-zero cuspform Δ is identified, a similar argument gives the structure of M_{2k}, for $2k \geq 4$ so that Eisenstein series converge. Namely, given $f \in M_{2k}$, subtract a multiple of E_{2k} to obtain a cuspform of weight $2k$, and then divide by Δ to obtain a modular form of weight $2k - 12$. This shows that $M_{2k} = \mathbb{C}E_{2k} + \Delta M_{2k-12}$, as claimed. ///

For present purposes, an *isobaric* polynomial $P(X, Y) \in \mathbb{C}[X, Y]$ (with weights $4, 6$) is a polynomial with the property that there is an integer $2k$ such that every monomial X^aY^b appearing has the property that $4a + 6b = 2k$. This has the effect that $P(E_4, E_6)$ is a modular form of weight 12.

[3.1.2] Corollary: Every holomorphic modular form for $SL_2(\mathbb{Z})$ is an isobaric polynomial in E_4, E_6.

Proof: The assertion is vacuously true for weight 0 since holomorphic modular forms of weight 0 are constants. Holomorphic modular forms of weight 2 are all identically 0. At weights 4 and 6, all modular forms are *multiples* of the respective Eisenstein series.

At weight 8, the only modular form is E_8, but also E_4^2 has weight 8. Both have 0^{th} Fourier coefficient 1, so $E_8 = E_4^2$. Similarly, $E_{10} = E_4 \cdot E_6$.

We already showed that Δ is a constant multiple of the isobaric polynomial $E_4^3 - E_6^2$. Since $E_{12} - E_4^3$ is a cuspform of weight 12, it is a multiple of Δ, proving that E_{12} has an isobaric polynomial expression in terms of E_4 and E_6.

Given $12 < 2k \in 2\mathbb{Z}$, find non-negative integers a, b such that $4a + 6b = 2k$. Then $E_{2k} - E_4^aE_6^b$ is a cuspform, and

$$\frac{E_{2k} - E_4^aE_6^b}{\Delta} \quad \in \quad M_{2k-12}$$

By induction, E_{2k} is an isobaric polynomial in E_4, E_6. Given $f \in M_{2k}$, subtract a multiple of E_{2k} to produce a cuspform f_2, allowing division by Δ to put f_2/Δ in M_{2k-12}, completing the induction. ///

[3.1.3] Corollary: For every weight $2k$, the space of holomorphic cuspforms is finite-dimensional.

Proof: The space of cuspforms of weight $2k$ is $\Delta \cdot M_{2k-12}$, and M_{2k-12} is cuspforms together with multiples of E_{2k-12}, for $2k - 12 \geq 4$. ///

[3.1.4] Remark: [Ramanujan 1916] conjectured that the n^{th} Fourier coefficient $\tau(n)$ of Δ satisfies

$$|\tau(p)| \quad \leq \quad 2p^{\frac{11}{2}} \qquad\qquad \text{(for prime } p\text{)}$$

and

$$\tau(mn) \quad = \quad \tau(m) \cdot \tau(n) \qquad\qquad \text{(for coprime } m, n\text{)}$$

and

$$\tau(p^{n+1}) \quad = \quad \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \qquad\qquad \text{(for prime } p\text{)}$$

[Mordell 1917] prove the latter two properties, using operators systematically investigated in [Hecke 1937], nowadays called *Hecke operators*. As noted earlier, [Deligne 1974] proved $|c_p| \le 2p^{k-\frac{1}{2}}$ for prime p and weight $2k$ cuspforms, as consequence of his completion of proof of the *Weil conjectures*.

[3.1.5] Remark: In this context, *the modular function* or *modular invariant* $j(z)$ is defined to be a constant multiple of E_4^3/Δ, since now we know that $\Delta \ne 0$ in $\mathfrak{H}$, and that Δ has a simple pole at $i\infty$ and vanishes at ρ since $E_4(\rho) = 0$.

[3.1.6] Remark: Yes, there is some conflict with the notation that j can refer to the *cocycle*, as well as to the modular function, but context usually clarifies.

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4. *Proof of divisor/dimension formula*

This proof of

$$\frac{n_i}{2} + \frac{n_\rho}{3} + n_{i\infty} + \sum_{\text{other } z} n_z \;=\; \frac{2k}{12}$$

is an application of the *argument principle*, exploiting the near-invariance of modular forms.

Proof: Let f be a not-identically-zero holomorphic modular form of weight $2k$. Let

$$F_T \;=\; \{|z| \ge 1,\ |\mathrm{Re}(z)| \le \tfrac{1}{2},\ \mathrm{Im}(z) \le T\}$$

be the truncation at height T of the standard fundamental domain F, and γ a path tracing its boundary.

On one hand, by the argument principle,

$$\int_\gamma \frac{f'(z)}{f(z)}\,dz \;=\; 2\pi i \sum_{z \text{ inside } F_T} \nu_f(z)$$

In fact, points on the boundary itself require special treatment, especially the points i and ρ. Treatment of this is postponed to the end of the proof.

On the other hand, the individual pieces of the path integral nearly cancel each other out, except for some manageable pieces, as follows.

The easiest part is that the integrals along the *upward* path along $\mathrm{Re}(z) = +\frac{1}{2}$ and *downward* path along $\mathrm{Re}(z) = -\frac{1}{2}$ *cancel* each other, because $f(z+1) = f(z)$.

Let $f(z) = \sum_{n \ge n_o} c_n\, e^{2\pi i n z}$, with $c_{n_o} \ne 0$. That is, $\nu_{i\infty}(f) = n_o$. The path-integral along the top of ∂F_T, from $\frac{1}{2} + iT$ to $-\frac{1}{2} + iT$ is an integral in the coordinate $q = e^{2\pi i n z}$ around a circle: letting $g(q) = f(z)$,

$$\int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{f'(x+iT)}{f(x+iT)}\,dx \;=\; \int_{\frac{1}{2}+iT}^{-\frac{1}{2}+iT} \frac{g'(q)\cdot \frac{dq}{dz}}{g(q)}\,dz \;=\; \int_C \frac{g'(q)}{g(q)}\,dq$$

with C a circle of radius $e^{-2\pi T}$ at 0, traced *clockwise*. *The Fourier expansion of f in z is a power series expansion in q*, so by the *argument principle*, and by the convention about $\nu_f(i\infty)$,

$$\int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{f'(x+iT)}{f(x+iT)}\,dx \;=\; -2\pi i \cdot \nu_f(i\infty) - 2\pi i \sum_{z:\mathrm{Im}(z) > T} \nu_f(z)$$

The path from the cube-root of unity ρ to i is mapped by $z \to -1/z$ to that running backward from the sixth root of unity to i, but these do not quite cancel each other, because f is not *invariant* under $z \to -1/z$. Rather, differentiating $f(-1/z) = z^{2k} \cdot f(z)$ gives

$$f'(-1/z) \cdot \frac{1}{z^2} \;=\; 2kz^{2k-1} f(z) + z^{2k} f'(z)$$

so

$$f'(-1/z) \;=\; 2kz^{2k+1}f(z) + z^{2k+2}f'(z)$$

and

$$\frac{f'(-1/z)}{f(-1/z)}d(-1/z) \;=\; \frac{2kz^{2k+1}f(z) + z^{2k+2}f'(z)}{z^{2k}f(z)}\frac{dz}{z^2} \;=\; \frac{2k}{z} + \frac{f'(z)}{f(z)}$$

Thus, the integral from the cube root of 1 to the sixth root of 1 cancel *except* for the $-2k/z$. Letting $z = e^{it}$ as t goes from $\frac{2}{3}\pi$ to $\frac{1}{2}\pi$,

$$\int_{\frac{2}{3}\pi}^{\frac{1}{2}\pi} \Big(\frac{f'(z)}{f(z)}\,dz \;-\; \frac{f'(-1/z)}{f(-1/z)}\Big)\,d(-1/z) \;=\; \int_{\frac{2}{3}\pi}^{\frac{1}{2}\pi} \frac{-2k}{e^{-it}}\,d(e^{it}) \;=\; \int_{\frac{2}{3}\pi}^{\frac{1}{2}\pi} -2ik\;dt \;=\; 2ik\cdot\frac{\pi}{6} \;=\; 2\pi i\cdot\frac{2k}{12}$$

Thus, *if there were no vanishing on the boundary*, evaluating the integral around the truncated fundamental domain in two ways gives

$$\sum_{z\ \mathrm{Im}\,(z)<T} \nu_f(z) \quad=\quad -\nu_f(i\infty) - \sum_{z:\mathrm{Im}\,(z)>T} \nu_f(z) \;+\; \frac{2k}{12}$$

or

$$\nu_f(i\infty) + \sum_{z\in F} \nu_f(z) \quad=\quad \frac{2k}{12}$$

Now we consider points on the boundary of F_T. Any vanishing on the top edge $\mathrm{Im}\,(z) = T$ can be avoided by adjusting T slightly. Any vanishing on the vertical edges $\mathrm{Re}\,(z) = \pm\frac{1}{2}$ can be easily accommodated by slightly deforming the contour γ *inward* on the *left* side $\mathrm{Re}\,(z) = -\frac{1}{2}$ to *exclude* a point z_o with $f(z_o) = 0$, and deforming the contour slightly *outward* on the right side $\mathrm{Re}\,(z) = \frac{1}{2}$ to *include* $z_o + 1$. Similarly, for any point on the bottom part of the boundary, except for i and ρ, at which f vanishes, the left half of that arc can be deformed slightly inward, and the right half outward, to avoid the points. [5] Thus, the ordinary argument principle is sufficient for these cases.

[4.1] Points i, ρ on the boundary

Unfortunately, there is no deformation of the contour to avoid the points i, ρ while counting order-of-vanishing. We first consider the situation at i.

To simplify the discussion, use the *Cayley map* $z \to \frac{z-i}{-iz+1}$ to convert the arc along $|z| = 1$ to a straight line segment σ along the real axis, and replace f by its composition g with the inverse $z \to \frac{z+i}{iz+1}$ to the Cayley map. This does not alter order-of-vanishing. In these coordinates modify σ traversing the interval $[-a, a]$ left-to-right to include a small semi-circular detour along $|z| = \varepsilon$ in the upper half-plane. That is, the modified path σ_ε goes along the interval $[-a, -\varepsilon]$ left-to-right, along the arc clockwise from $-\varepsilon$ to $+\varepsilon$, and left-to-right along the interval $[\varepsilon, a]$.

For $g(0) = 0$, the logarithmic derivative g'/g has a simple pole at 0, with Laurent expansion

$$\frac{g'(z)}{g(z)} \;=\; \frac{\nu_0(g)}{z} + (\text{holomorphic near } 0)$$

By continuity, the limit as $\varepsilon \to 0^+$ of the integral of a holomorphic function along the modified paths σ_ε is just the integral along the segment σ. This leaves us explicit computation of

$$\int_{\sigma_\varepsilon} \frac{dz}{z} \;=\; \int_{-a}^{-\varepsilon} \frac{dt}{t} + \int_{-\pi}^{0} \frac{d(\varepsilon e^{it})}{e^{it}} \int_{\varepsilon}^{a} \frac{dt}{t} \;=\; -(\log a - \log\varepsilon) - \pi i + (\log a - \log\varepsilon) \;=\; -\pi i$$

[5] One might reasonably worry that there might be infinitely-many points near F_T where f vanishes. However, the *compactness* of any slightly larger region containing F_T, and the holomorphy of f, assures that this cannot happen.

That is, the limit of the integrals over paths σ_ε *excluding* 0 produces $\frac{1}{2} \cdot 2\pi i \cdot \nu_g(0)$. Thus, the corresponding modification of the path around the boundary of F_T gives $-\frac{1}{2} \cdot 2\pi i \cdot \nu_f(i)$.

The point ρ is treated similarly, with slight further complications. One way to describe the outcome is to treat ρ and $\rho+1$ separately, as follows. Here, unlike at i, we cannot completely convert the path near ρ into straight line segments. Nevertheless, there is a well-defined angle to the boundary of F at ρ, namely, $\pi/3$. Modifying the path-integral along the boundary by indenting upward along a small arc of radius $\varepsilon > 0$, and taking a limit as $\varepsilon \to 0^+$, produces $-\frac{1}{6} \cdot 2\pi i \cdot \nu_f(\rho)$, rather than the full $-2\pi i \cdot \nu_f(\rho)$. Similarly, the limit of slightly-indented paths around $\rho+1$ produces another $-\frac{1}{6} \cdot 2\pi i \cdot \nu_f(\rho)$, noting that $\nu_f(\rho+1) = \nu_f(\rho)$.

Thus, by integrating over the boundary of F_T modified by indentations of radius ε at i and ρ, and taking the limit as $\varepsilon \to 0^+$, we obtain

$$\nu_f(i\infty) + \sum_{z\in F} \nu_f(z) \quad = \quad -\frac{\nu_f(i)}{2} - \frac{\nu_f(\rho)}{3} + \frac{2k}{12}$$

Moving the suitably weighted orders of vanishing at i, ρ to the left-hand side gives the divisor/dimension formula. ///

[4.1.1] **Remark:** The idea that path integrals essentially running directly *through* a simple pole can be construed as giving *half* the residue, or half the negative, depending on the direction of indentation, can be legitimized as in the discussion of i above. The further idea, applied above to ρ and $\rho+1$, that path integrals along paths having a *corner* with angle θ at a simple pole, can be construed as producing $-\frac{\theta}{2\pi}$ of the residue, can likewise be legitimized. In all these cases, the underlying mechanism is that

$$\int_{\theta_1}^{\theta_2} \frac{d(\varepsilon e^{it})}{\varepsilon e^{it}} \quad = \quad \int_{\theta_1}^{\theta_2} i\, dt \quad = \quad (\theta_2 - \theta_1) i \qquad\qquad \text{(independent of } \varepsilon > 0\text{)}$$

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5. *Fourier expansions of holomorphic Eisenstein series*

[5.0.1] **Theorem:** For weight $2k \geq 4$, the holomorphic Eisenstein series

$$E_{2k}(z) \;=\; \sum_{\text{coprime } c,d} \frac{1}{cz+d}^{2k}$$

has Fourier expansion

$$E_{2k}(z) \quad = \quad 1 \;+\; \frac{(-2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)} \sum_{n\geq 1} \sigma_{2k-1}(n)\, e^{2\pi i n z}$$

Before the important computation that determines the Fourier coefficients, two corollaries:

[5.0.2] **Corollary:** Given a modular form $f(z) = c_o + \sum_{n\geq 1} c_n\, e^{2\pi i n z}$, the difference $f - c_o \cdot E_{2k}$ is a *cuspform*.

Proof: The leading Fourier coefficient of the Eisenstein series is 1, so the indicated subtraction exactly annihilates the leading Fourier coefficient. ///

[5.0.3] **Corollary:** For weight $2k \geq 4$, the holomorphic Eisenstein series $E_{2k}(z)$ is *bounded* in the standard fundamental domain, so is a elliptic modular form in the strongest sense.

Proof: The absence of negative-index Fourier terms, and an easy estimate

$$\sigma_{2k-1}(n) \;\leq\; \sum_{1\leq \ell\leq n} \ell^{2k-1} \;\leq\; (n+1)^{2k} \;\ll\; e^{2\pi n} \qquad\qquad \text{(as } n \to +\infty\text{)}$$

give

$$|E_{2k}(z) \ll 1 + \sum_{n\geq 1} e^{2\pi n}\, e^{-2\pi n y} \leq 1 + \frac{e^{-2\pi y}}{1 - e^{-2\pi y}}$$

which is bounded for $y \geq \frac{\sqrt{3}}{2}$. ///

Proof: We directly compute the Fourier coefficients

$$c_n = e^{2\pi n y} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x}\, E^*_{2k}(x+iy)\, dx$$

of the renormalized Eisenstein series

$$E^*_{2k}(z) = \zeta(2k)\cdot E_{2k}(z) = \sum_{(c,d)\neq(0,0)} \frac{1}{(cz+d)^{2k}}$$

First, the subsum over $d \neq 0$ with $c = 0$ is literally $2\zeta(2k)$, and this is translation-invariant, so is part of the 0^{th} Fourier coefficient 0

Each subsum over $d \in \mathbb{Z}$ for fixed $c \neq 0$ is invariant under $z \to z+1$, so has a Fourier expansion, with n^{th} coefficient

$$e^{2\pi n y} \cdot \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} \sum_d \frac{1}{(cz+d)^{2k}}\, dx$$

The integral is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} \sum_d \frac{1}{(cx+d+ciy)^{2k}}\, dx = c^{-2k} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} \sum_d \frac{1}{(x+\frac{d}{c}+iy)^{2k}}\, dx$$

Aiming to *unwind* the sum-and-integral to have a simpler sum and an integral over $\mathbb{R}$, rewrite

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} \sum_d \frac{1}{(x+\frac{d}{c}+iy)^{2k}}\, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} \sum_{\ell\in\mathbb{Z}} \sum_{d \bmod c} \frac{1}{(x+\ell+\frac{d}{c}+iy)^{2k}}\, dx$$

and replace x by $x - \ell$, to obtain

$$\sum_{\ell\in\mathbb{Z}} \int_{-\frac{1}{2}+\ell}^{\frac{1}{2}+\ell} e^{-2\pi i n x} \sum_{d \bmod c} \frac{1}{(x+\frac{d}{c}+iy)^{2k}}\, dx = \int_{\mathbb{R}} e^{-2\pi i n x} \sum_{d \bmod c} \frac{1}{(x+\frac{d}{c}+iy)^{2k}}\, dx$$

$$= \sum_{d \bmod c} \int_{\mathbb{R}} e^{-2\pi i n x} \frac{1}{(x+\frac{d}{c}+iy)^{2k}}\, dx = \sum_{d \bmod c} e^{2\pi i n d/c} \int_{\mathbb{R}} e^{-2\pi i n x} \frac{1}{(x+iy)^{2k}}\, dx$$

by replacing x by $x - \frac{d}{c}$ in each integral. Now neither c nor d appears inside the integral, while neither x nor y appear in the sum.

The integral can be evaluated by residues, treating x itself as a complex variable, as follows. Fix y, the imaginary part of the original z. For $n \leq 0$, the function $e^{2\pi i n x}$ is rapidly decreasing as x moves into the upper half-plane, so the indicated integral is the limit as $R \to +\infty$ of an integral left-to-right along $[-R, R]$ and then along an arc of a circle of radius R in the upper half-plane. This picks up residues of $x \to e^{-2\pi i n x}/(x+iy)^{2k}$ in the upper half-plane: there are none, so these Fourier coefficients are 0.

For $n > 0$, the integral can be evaluated by residues, using an arc of a circle in the *lower* half-plane, picking up $-2\pi i$ times the residue of $x \to e^{-2\pi i n x}/(x+iy)^{2k}$ at $-iy$, namely,

$$\frac{-2\pi i}{(2k-1)!} \cdot \Big(\frac{\partial}{\partial x}\Big)^{2k-1} e^{-2\pi i n x}\Big|_{x=-iy} = \frac{-2\pi i}{(2k-1)!} \cdot (-2\pi i n)^{2k-1} \cdot e^{-2\pi n y} = \frac{(2\pi i)^{2k}}{(2k-1)!}\, n^{2k-1}\, e^{-2\pi n y}$$

That is,

$$\int_{\mathbb{R}} e^{-2\pi inx} \frac{1}{(x+iy)^{2k}}\,dx \;=\; \begin{cases} \dfrac{(2\pi i)^{2k}}{(2k-1)!}\, n^{2k-1}\, e^{-2\pi ny} & (\text{for } n \ge 1) \\ 0 & (\text{for } n \le 0) \end{cases}$$

The sum over $d \bmod c$ is a sum of the *character* $d \to e^{2\pi ind/c}$ over the finite abelian group $\mathbb{Z}/c$. The *cancellation lemma* says this sum is 0 unless the character is *trivial*, in which case it is the cardinality of the group, namely, $|c|$. The character is trivial if and only $c|n$. Thus,

$$\sum_{d \bmod c} e^{2\pi ind/c} \;=\; \begin{cases} |c| & (\text{for } c|n) \\ 0 & (\text{otherwise}) \end{cases}$$

In summary, the 0^{th} Fourier coefficient is $2\zeta(2k)$, the negative-index Fourier coefficients are 0, and for $n > 1$ the Fourier coefficient is

$$\sum_{c|n} \frac{1}{c^{2k}} \cdot |c| \;\times\; \frac{(2\pi i)^{2k}}{(2k-1)!}\, n^{2k-1} \qquad\qquad (\text{for } n > 1)$$

As c runs over positive and negative divisors of n, so does n/c, and the last expression can be simplified somewhat by doing so:

$$\sum_{c|n} \frac{c^{2k}}{n^{2k}} \left|\frac{n}{c}\right| \times \frac{(2\pi i)^{2k}}{(2k-1)!}\, n^{2k-1} \;=\; \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{0<c|n} c^{2k-1}$$

Often the sum of ℓ^{th} powers of positive divisors of an integer n is denoted $\sigma_\ell(n)$, so the Fourier expansion of the Eisenstein series can be written

$$2\zeta(2k)\cdot E_{2k}(z) \;=\; 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!} \sum_{n\ge 1} \sigma_{2k-1}(n)\, e^{2\pi inz}$$

and

$$E_{2k}(z) \;=\; 1 \;+\; \frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)} \sum_{n\ge 1} \sigma_{2k-1}(n)\, e^{2\pi inz}$$

as claimed. ///

[5.0.4] Corollary: $E_4^2 = E_8$, $E_4E_6 = E_{10}$, and $E_4E_{10} = E_6E_8 = E_{14}$.

Proof: In dimensions $8, 10, 14$ there are no holomorphic modular forms other than the corresponding Eisenstein series, and the leading Fourier coefficients are always 1. ///

[5.0.5] Corollary: Granting that $\zeta(2k)$ is a rational multiple of π^{2k}, the Fourier coefficients of Eisenstein series are *rational numbers.* ///

[5.0.6] Remark: The rationality of the Fourier coefficients of holomorphic Eisenstein series is significant in later developments. The following corollaries are slightly frivolous examples of proving number-theoretic identities by relations among automorphic forms. Nevertheless, more serious results do use the same proof mechanism of which these simple examples are prototypes.

[5.0.7] Corollary: For positive integers N,

$$\sigma_7(N) \;=\; 2\cdot\frac{7!\,\zeta(8)}{3!\,(2\pi i)^4\,\zeta(4)}\sigma_3(N) + \frac{7!\,\zeta(8)}{(3!)^2\,\zeta(4)^2}\sum_{m+n=N}\sigma_3(m)\,\sigma_3(n) \qquad (\text{with } m,n\ge 1)$$

$$\sigma_9(N) \;=\frac{9!\,\zeta(10)}{3!\,(2\pi i)^6\,\zeta(4)}\sigma_3(N) + \frac{9!\,\zeta(10)}{5!\,(2\pi i)^4\,\zeta(6)}\sigma_5(N) + \frac{9!\,\zeta(10)}{3!\,5!\,\zeta(4)\,\zeta(6)}\sum_{m+n=N}\sigma_3(m)\,\sigma_5(n) \qquad (m,n\ge 1)$$

Proof: The first identity comes from equating the Fourier coefficients of $E_4^2 = E_8$. A similar one arises from $E_4E_6 = E_{10}$. Fourier expansions without negative-index terms multiply as

$$\sum_{m\ge 0} a_m\, e^{2\pi i m z}\;\cdot\sum_{n\ge 0} b_m\, e^{2\pi i n z} \;=\; \sum_{N\ge 0}\Big(\sum_{m+n=N} a_m\cdot b_n\Big)e^{2\pi i N z}$$

From $E_4^2 = E_8$, noting that the 0^{th} Fourier coefficients do not quite fit into the general pattern, for $N\ge 1$, equating the N^{th} coefficients of E_4^2 and E_8 gives

$$\frac{(2\pi i)^8}{7!\,\zeta(8)}\sigma_7(N) \;=\; 2\cdot\frac{(2\pi i)^4}{3!\,\zeta(4)}\sigma_3(N) \;+\; \Big(\frac{(2\pi i)^4}{3!\,\zeta(4)}\Big)^2\sum_{m+n=N}\sigma_3(m)\,\sigma_3(n)$$

Rearranging,

$$\sigma_7(N) \;=\; 2\cdot\frac{7!\,\zeta(8)}{3!\,(2\pi i)^4\,\zeta(4)}\sigma_3(N) + \frac{7!\,\zeta(8)}{(3!)^2\,\zeta(4)^2}\sum_{m+n=N}\sigma_3(m)\,\sigma_3(n)$$

The second computation is entirely analogous. ///

[5.0.8] Remark: Also, these frivolous relations completely determine $\zeta(4)$, $\zeta(6)$, $\zeta(8)$, and $\zeta(10)$, by looking at the relations for $N = 1, 2$. And since there are no cuspforms of weight 14, also $\zeta(14)$ is determined.

More generally, from [Gunning 1959/62] p. 55, Ramanujan proved the following, but with a worse error term, since Hecke's estimate on Fourier coefficients of cuspforms was not available. That is, in general, $E_{2k}\cdot E_{2\ell}$ is probably not exactly $E_{2k+2\ell}$, but it misses only by a cuspform:

[5.0.9] Corollary: For $2k\ge 4$ and $2\ell\ge 4$ and $N\ge 1$,

$$\sigma_{2k+2\ell-1}(N) \;=\; \frac{(2k+2\ell-1)!\,\zeta(2k+2\ell)}{(2\pi i)^{2\ell}\,(2k-1)!\,\zeta(2k-1)}\,\sigma_{2k-1}(N) + \frac{(2k+2\ell-1)!\,\zeta(2k+2\ell)}{(2\pi i)^{2k}\,(2\ell-1)!\,\zeta(2\ell)}\,\sigma_{2\ell-1}(N)$$

$$+\;\frac{(2k+2\ell-1)!\,\zeta(2k+2\ell)}{(2k-1)!\,(2\ell-1)!\,\zeta(2k)\,\zeta(2\ell)}\sum_{m+n=N}\sigma_{2k-1}(m)\cdot\sigma_{2\ell-1}(m) \;+\; O(n^{\frac{2k+2\ell}{2}}) \qquad (\text{with } m,n\ge 1)$$

Proof: Up to a cuspform, $E_{2k}\cdot E_{2\ell} = E_{2k+2\ell}$. Equating the N^{th} Fourier coefficients and multiplying through by $(2k+2\ell-1)!\,\zeta(2k+2\ell)/(2\pi i)^{2k+2\ell}$ gives the identity, with the big-O term arising from Hecke's estimate on the Fourier coefficients of the cuspform $= E_{2k+2\ell} - E_{2k}\cdot E_{2\ell}$. ///

[5.0.10] Remark: Of course, for weights $2k+2\ell$ among $8, 10, 14$, there are no cuspforms, and the error term is exactly 0.

6. *Automorphic L-functions*

[6.1] Euler product attached to $\Delta(z)$ A little later, we will prove two of the conjectures of Ramanujan proven by Mordell, in a form applicable to all holomorphic cuspforms of for $SL_2(\mathbb{Z})$. First, we examine the implications for Dirichlet series.

With $\Delta(z) = 1 \cdot e^{2\pi i n z} + \sum_{n\geq 1} \tau(n)\, e^{2\pi i n z}$ the unique cuspform of weight 12 for $SL_2(\mathbb{Z})$, the associated *Dirichlet series* is

$$L(s,\Delta) \;=\; \sum_{n\geq 1} \frac{\tau(n)}{n^s}$$

The Hecke estimate $|\tau(n)| \ll n^{\frac{12}{2}}$ shows that the series for $L(s,\Delta)$ is absolutely convergent for $\mathrm{Re}(s) > \frac{12}{2}+1$.

The *weak multiplicativity* $\tau(mn) = \tau(m)\cdot\tau(n)$ for coprime m,n is equivalent to an *Euler factorization* of $L(s,\Delta)$:

$$L(s,\Delta) \;=\; \sum_{n\geq 1} \frac{\tau(n)}{n^s} \;=\; \prod_{p \text{ prime}} \Big(1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{p^{2s}} + \frac{\tau(p^3)}{p^{3s}} + \ldots\Big)$$

The more peculiar relation

$$\tau(p^{n+1}) \;=\; \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1}) \qquad\qquad (\text{for prime } p, \text{ for } n\geq 1)$$

gives a *recursion* for the $\tau(p^n)$: to simplify notation, let $X = p^{-s}$, observe that powers of p^{-s} do multiply like powers of X, and

$$1\cdot\tau(p^{n+1})X^{n+1} \;-\; \tau(p)X\cdot\tau(p^n)X^n \;+\; p^{11}X^2\cdot\tau(p^{n-1})X^{n-1} \;=\; 0 \qquad\qquad (\text{for } n\geq 1)$$

For $n \geq 1$, the left-hand side of the last equality is the $X^{n+1^{th}}$ term in

$$\Big(1-\tau(p)X + p^{11}X^2\Big)\Big(1+\tau(p)X + \tau(p^2)X^2 + \tau(p^3)X^3 + \ldots\Big)$$

The constant component of the latter product is 1. That is,

$$\Big(1-\tau(p)X + p^{11}X^2\Big)\Big(1+\tau(p)X + \tau(p^2)X^2 + \tau(p^3)X^3 + \ldots\Big) \;=\; 1$$

That is,

$$\Big(1 - \frac{\tau(p)}{p^s} + \frac{p^{11}}{p^{2s}}\Big)\Big(1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{p^{2s}} + \frac{\tau(p^3)}{p^{3s}} + \ldots\Big) \;=\; 1$$

and

$$1 + \frac{\tau(p)}{p^s} + \frac{\tau(p^2)}{p^{2s}} + \frac{\tau(p^3)}{p^{3s}} + \ldots \;=\; \frac{1}{1 - \dfrac{\tau(p)}{p^s} + \dfrac{p^{11}}{p^{2s}}}$$

Thus,

$$\sum_n \frac{\tau(n)}{n^s} \;=\; \prod_p \frac{1}{1 - \dfrac{\tau(p)}{p^s} + \dfrac{p^{11}}{p^{2s}}}$$

This Euler product factorization partly justifies calling $\sum_n \frac{\tau(n)}{n^s}$ an *automorphic L-function.*

Further, the discriminant of the quadratic equation

$$X^2 - \tau(p)X + p^{11} \;=\; 0$$

is $\tau(p)^2 - 4p^{11}$. From the expression of Δ as a real constant multiple of $E_4^6 - E_6^2$, $\tau(p) \in \mathbb{R}$. Thus, the roots occur in complex conjugate pairs exactly when Ramanujan's conjectured, Deligne's proven, inequality $|\tau(p)| < 2p^{\frac{11}{2}}$ holds.

[6.1.1] Remark: We have given Hecke's proof of $|\tau(p)| \ll p^{\frac{12}{2}}$, but will not attempt to follow [Deligne 1974] to prove $|\tau(p)| < 2p^{\frac{11}{2}}$.

[6.1.2] Remark: We will show below that the space of weight $2k$ holomorphic cuspforms for $SL_2(\mathbb{Z})$ has a basis of cuspforms $f(z) = \sum_{n\geq 1} c_n\, e^{2\pi i n z}$ with $c_n = 1$ and whose associated Dirichlet series

$$L(s,f) \;=\; \sum_{n\geq 1} \frac{c_n}{n^s}$$

have Euler product factorizations

$$L(s,f) \;=\; \sum_{n\geq 1} \frac{c_n}{n^s} \;=\; \prod_p \frac{1}{1 - \dfrac{c_p}{p^s} + \dfrac{p^{2k-1}}{p^{2s}}}$$

Having an Euler product partly justifies calling $L(s,f)$ an *automorphic L-function* attached to f. The Hecke estimate $c_n \ll n^{\frac{2k}{2}}$ proves absolute convergence of $L(s,f)$ for $\mathrm{Re}(s) > \frac{2k}{2} + 1$.

[6.2] Analytic continuation and functional equation

A holomorphic cuspform $f(z) = \sum_{n\geq 1} c_n\, e^{2\pi i n z}$ of weight $2k$ for $SL_2(\mathbb{Z})$ has associated Dirichlet series

$$L(s,f) \;=\; \sum_{n\geq 1} \frac{c_n}{n^s}$$

whether or not this has an Euler product.

[6.2.1] Remark: Merely copying Fourier coefficients to coefficients of a Dirichlet series accomplishes little, without further analytic features.

We do know that f is *rapidly decreasing* as $y \to +\infty$, and that $y^{\frac{2k}{2}} \cdot |f(z)|$ is *bounded* on $\mathfrak{H}$, so $|f(z)| \ll y^{-k}$ as $y \to 0^+$. Thus, for $\mathrm{Re}(s) > k$ we have absolute convergence of the *Mellin transform*

$$\int_0^\infty y^s\, f(iy)\, \frac{dy}{y}$$

In that range,

$$\int_0^\infty y^s\, f(iy)\, \frac{dy}{y} \;=\; \int_0^\infty y^s \sum_n c_n\, e^{-2\pi n y}\, \frac{dy}{y} \;=\; \sum_n c_n \int_0^\infty y^s\, e^{-2\pi n y}\, \frac{dy}{y}$$

$$= \sum_n \frac{c_n}{(2\pi n)^s} \cdot \int_0^\infty y^s\, e^{-y}\, \frac{dy}{y} \;=\; (2\pi)^{-s}\,\Gamma(s) \sum_n \frac{c_n}{n^s} \;=\; (2\pi)^{-s}\,\Gamma(s)\, L(s,f)$$

[6.2.2] Claim: $(2\pi)^{-s}\,\Gamma(s)\,L(s,f)$ has an analytic continuation to an *entire* function, satisfying

$$(2\pi)^{-(2k-s)}\,\Gamma(2k-s)\,L(2k-s,f) \;=\; (-1)^{\frac{2k}{2}} \cdot (2\pi)^{-s}\,\Gamma(s)\,L(s,f)$$

[6.2.3] Remark: This *integral representation* of $L(s,f)$, with Gamma-factor $(2\pi)^{-s}\,\Gamma(s)$ to *complete* it, plays the role for $L(s,f)$ as did the integral representation of the completed $\zeta(s)$ in terms of $\theta(z)$.

[6.2.4] Remark: With hindsight, seeing that the functional equation is with respect to $s \leftrightarrow 2k - s$, a contemporary choice would be to renormalize to have a functional equation $s \leftrightarrow 1 - s$, as we describe below. The latter convention is *not* universal.

Proof: The rapid decay of a cuspform $f(x + iy)$ as $y \to +\infty$ assures that part of the integral is *entire*:

$$\int_1^\infty y^s\, f(iy)\, \frac{dy}{y} \;=\; \text{entire}$$

Meanwhile, using the automorphy condition with $z \to -1/z$,

$$\int_0^1 y^s\, f(iy)\, \frac{dy}{y} \;=\; \int_0^1 y^s\, (iy)^{-2k} \cdot f(-1/iy)\, \frac{dy}{y} \;=\; (-1)^{\frac{2k}{2}} \int_0^1 y^{s-2k} \cdot f(-1/iy)\, \frac{dy}{y}$$

$$= \; (-1)^{\frac{2k}{2}} \int_1^\infty y^{2k-s} \cdot f(iy)\, \frac{dy}{y} \;=\; \text{entire}$$

Thus,

$$(2\pi)^{-s}\, \Gamma(s)\, L(s, f) \;=\; \int_1^\infty y^s\, f(iy)\, \frac{dy}{y} \;+\; (-1)^{\frac{2k}{2}} \int_1^\infty y^{2k-s}\, f(iy)\, \frac{dy}{y} \;=\; \text{entire}$$

and the behavior under $s \leftrightarrow 2k - s$ is clear. ///

[6.2.5] Remark: To translate so that the functional equation is $s \leftrightarrow 1 - s$, instead of the natural but naive normalization above, put

$$L(s, f) \;=\; \sum_n \frac{c_n/n^{\frac{2k-1}{2}}}{n^s} \;=\; \sum_n \frac{c_n}{n^{s+\frac{2k-1}{2}}}$$

The corresponding integral representation becomes

$$(2\pi)^{-s-\frac{2k-1}{2}}\, \Gamma(s + \frac{2k-1}{2})\, L(s, f) \;=\; \int_0^\infty y^{s-\frac{1}{2}} \Big(f(iy) \cdot y^{\frac{2k}{2}} \Big) \frac{dy}{y}$$

Then one might further divide through by a constant so that the extra constant power of π disappears, giving functional equation

$$(2\pi)^{-(1-s)}\, \Gamma(1 - s + \frac{2k-1}{2})\, L(1 - s, f) \;=\; (-1)^k \cdot (2\pi)^{-s}\, \Gamma(s + \frac{2k-1}{2})\, L(s, f)$$

[6.2.6] Remark: Thus, we have shown that automorphic L-functions $L(f, s)$ arising from holomorphic cuspforms for $SL_2(\mathbb{Z})$ have analytic continuations and functional equations. Euler product factorizations are proven below.

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