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Divergence of Fourier series of C^o functions

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http://www.math.umn.edu/˜garrett/m/complex/notes_2014-15/11a_divergence_fourier.pdf]

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1. *Divergence of Fourier series of C^o functions*

Convergence of Fourier series means *convergence of the sequence of partial sums*, for any type of convergence, whether *pointwise*, *uniformly pointwise*, or L^2.

The *density* of finite Fourier series in $C^o(S^1)$ makes no claim about *which* finite Fourier series approach a given $f \in C^o(S^1)$. Indeed, the density proof given via the Féjer kernel uses finite Fourier series quite distinct from the finite partial sums of the Fourier series of f itself, namely,

$$N^{th} \text{ Féjer sum } = \frac{1}{2N+1} \sum_{|n|\le N} (2N+1-|n|) \cdot \widehat{f}(n) \cdot e^{inx}$$

As another failure: the general discussion of L^2 functions shows that a Cauchy sequence of L^2 functions has a *subsequence* converging *pointwise*. Indeed, this proves existence of the limit, to prove completeness of L^2. This applies to Fourier series, but does *not* say anything about the pointwise convergence of the *whole* sequence of partial sums, and does not address *uniformity* of the pointwise convergence.

The negative result here is a corollary of the Banach-Steinhaus theorem for Banach spaces.

[1.0.1] Corollary: There is $f \in C^o(S^1)$ whose Fourier series

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{inx} \qquad\qquad (\text{with } \widehat{f}(n) = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} e^{-inx} f(x)\,dx)$$

diverges at 0. In fact, the divergence can be arranged at any given countable set of points on S^1.

Proof: To invoke Banach-Steinhaus, consider the functionals given by partial sums of the Fourier series of f, evaluated at 0:

$$\lambda_N(f) = \sum_{|n|\le N} \hat{f}(n) = \sum_{|n|\le N} \hat{f}(n) \cdot e^{in\cdot 0}$$

There is an easy upper bound

$$|\lambda_N(f)| \le \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{|n|\le N} e^{-inx}\Big| \cdot \big|f(x)\big|\,dx \le |f|_{C^o} \cdot \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{|n|\le N} e^{-inx}\Big|\,dx = |f|_{C^o} \cdot \Big|\sum_{|n|\le N} e^{-inx}\Big|_{L^1(S^1)}$$

We will show that equality holds, namely, that

$$|\lambda_N| = \Big|\sum_{|n|\le N} e^{-inx}\Big|_{L^1}$$

and show that the latter L^1-norms go to ∞ as $N \to \infty$.

Summing the finite geometric series and rearranging:

$$\sum_{|n|\le N} e^{-inx} = \frac{e^{-iNx} - e^{-i(-N-1)x}}{e^{-ix}-1} = \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix}-e^{-ix}} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

The elementary inequality $|\sin t| \le |t|$ gives a lower bound

$$\int_0^{2\pi} \left|\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}\right| \, dx \;\ge\; \int_0^{2\pi} \left|\sin(N+\tfrac{1}{2})x\right| \cdot \frac{2}{x} \, dx \;=\; \int_0^{2\pi(N+\frac{1}{2})} |\sin x| \cdot \frac{2}{x} \, dx$$

$$\ge \sum_{\ell=1}^{N} \frac{1}{\ell} \int_{2\pi(\ell-1)}^{2\pi\ell} |\sin x| \, dx \;\ge\; \sum_{\ell=1}^{N} \frac{1}{\ell} \;\longrightarrow\; +\infty \qquad\qquad (\text{as } N \to \infty)$$

Thus, the L^1-norms do go to ∞.

We claim that the norm of the *functional* is the L^1-norm of the *kernel:* let $g(x)$ be the *sign* of the Dirichlet kernel

$$\sum_{|n|\le N} e^{-inx} \;=\; \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Let g_j be a sequence of periodic continuous functions with $|g_j| \le 1$ and going to g pointwise. [1] By *dominated convergence*

$$\lim_j \lambda_N(g_j) \;=\; \lim_j \int_0^{2\pi} g_j(x) \sum_{|n|\le N} e^{-inx} \, dx = \int_0^{2\pi} g(x) \sum_{|n|\le N} e^{-inx} \, dx = \int_0^{2\pi} |\sum_{|n|\le N} e^{-inx}| \, dx$$

By Banach-Steinhaus for the Banach space $C^o(S^1)$, since (as demonstrated above) there is *no* uniform bound $|\lambda_N| \le M$ for all N, there *exists* f in the unit ball of $C^o(S^1)$ such that

$$\sup_N |\lambda_N \, v| \;=\; +\infty$$

In fact, the collection of such v is *dense* in the unit ball, and is an intersection of a *countable* collection of dense open sets (a G_δ). That is, the Fourier series of f does not converge at 0.

The result can be strengthened by using Baire's theorem again. For a dense countable set of points x_j in the interval, let $\lambda_{j,N}$ be the continuous linear functionals on $C^o(\mathbb{R}/\mathbb{Z})$ defined by evaluation of finite partial sums of the Fourier series at x_j's:

$$\lambda_{j,N}(f) \;=\; \sum_{|n|\le N} \hat{f}(n) \, e^{inx_j}$$

As in the previous argument proof, the set E_j of functions f where

$$\sup_N |\lambda_{j,N} f| \;=\; +\infty$$

is a dense G_δ, so the intersection $E = \bigcap_j E_j$ is a dense G_δ, and, in particular, not empty. ///

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[1] The g_j will surely *not* converge *uniformly* pointwise.