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00. Complex numbers

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Complex analysis is one of the most natural and productive continuations of basic calculus, *not* addressing pathologies and pitfalls, but, instead, showing that natural functions have even better properties than imagined. The first 150 years of calculus addressed such nice functions, discovering the remarkable usefulness of calculus-as-complex-analysis implicitly, long before anyone worried about the subtleties and troubles highlighted in the 19th century. Euler, Lagrange, and the most effective of their contemporaries, inadvertently thought in terms we can now reinterpret as part of *complex analysis*, by often considering *function* to mean *expressible as (convergent?) power series*.

Complex numbers such as $\sqrt{-1}$ entered *not* by a perceived need to solve *quadratic* equations like $x^2+1=0$ lacking real-number roots, since such equations could simply be declared to have *no* roots, and forget about the whole difficulty. For that matter, *negative* numbers were considered merely convenient fictions by many mathematicians until just a few hundred years ago. Rather, an unavoidable awkwardness forced consideration of complex numbers, namely, the expression in radicals for roots of cubic equations $x^3+bx+c=0$ is something like

$$\text{roots} \;=\; \frac{\lambda+\lambda'}{3},\;\; \frac{\omega\lambda+\omega^2\lambda'}{3},\;\; \frac{\omega^2\lambda+\omega\lambda'}{3}$$

where

$$\lambda \;=\; \sqrt[3]{3bc-\frac{27}{2}c+(3\omega-\tfrac{1}{2})\sqrt{\Delta}} \qquad \lambda' \;=\; \sqrt[3]{3bc-\frac{27}{2}c+(3\omega^2-\tfrac{1}{2})\sqrt{\Delta}}$$

$$\omega \;=\; \frac{-1+\sqrt{-3}}{2} \;=\; \text{cube root of unity}$$

and, letting α,β,γ be the roots of the equation,

$$\Delta \;=\; \text{discriminant of cubic} \;=\; (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2 \;=\; -27c^2-4b^3$$

The awkwardness is that the cubic can have three *real* roots, thus inarguably legitimate, but the expression for them requires complex numbers.

Although Abel and Jacobi's work on elliptic integrals and elliptic functions (discussed later) necessarily refered to complex numbers, Cauchy's work on the basic calculus of complex analysis from the 1820s on showed its unexpected power, and made it indispensable to physicists and electrical engineers. Also, Cauchy argued for the genuine-ness of complex numbers by promoting the now-familiar pictorial representation as two-dimensional numbers corresponding to points in the plane, just as real numbers are construed as corresponding to points on the line. Indeed, Hamilton's later creation of *quaternions* was substantially motivated by an impulse to identify higher-dimensional numbers. Cauchy's basic complex analysis, as refined by Goursat, will be the first substantive body of results discussed, after some set-up.

Riemann's 1858/9 paper on the *zeta function* $\zeta(s)=\sum_{n\ge 1}\frac{1}{n^s}$ showed that complex analysis was definitely *not* just a different fashion for doing multi-variable calculus, as one might insist on viewing Cauchy's work. Specifically, beginning with Euler's discovery of the factorization

$$\zeta(s)=\sum_{n\ge 1}\frac{1}{n^s} \;=\; \prod_{p\text{ prime}}\frac{1}{1-p^{-s}} \qquad\qquad (\text{for } \mathrm{Re}(s)>1)$$

(proven by expanding each $1/(1-p^{-s})$ as a geometric series, multiplying these all together, and using *unique factorization*). Euler had already observed that this factorization gives a proof of the infinitude of primes more quantitative than the Euclidean proof, namely, that the sum expression blows up like $1/(s-1)$ as

$s \to 1^+$, so the product must do likewise. Riemann sketched an argument that (in von Mangoldt's modified form) the sum of $\log p$ for prime powers p^m below a bound T is *exactly* expressible in terms of the *complex* zeros ρ of $\zeta(s)$:

$$\sum_{\text{prime power } p^m < T} \log p \;=\; T \;-\; c \;-\; \sum_{\text{complex } \rho \,:\, \zeta(\rho)=0} \frac{T^\rho}{\rho} \;+\; \sum_{n\ge 1} \frac{T^{-2n}}{2n}$$

with a concrete but irrelevant constant c. Indeed, the analytic continuation of $\zeta(s)$ to the whole complex plane $\mathbb{C}$ vanishes at the negative even integers $-2n$, and these are the *trivial* zeros of $\zeta(s)$.

This *exact equality*, known as *Riemann's Explicit Formula*, is of a completely different nature than the merely asymptotic assertion in the *Prime Number Theorem*, with or without error estimates, that

$$\frac{\text{number of primes} < T}{T/\log T} \;\longrightarrow\; 1 \qquad\qquad (\text{as } T \to +\infty)$$

In fact, the latter far weaker result itself, although conjecture prior to 1800, was only proven 40 years *after* Riemann's paper, by Hadamard and de la Vallée Poussin, independently.

Indeed, filling out and legitimizing the arguments sketched in Riemann's paper was a central motivation in much of the development of complex analysis through the late 19th century and into the 20th.
