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02. Power series

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http://www.math.umn.edu/˜garrett/m/complex/notes_2020-21/02_power_series.pdf]

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1. *Convergence of power series*

The point is that power series $\sum_{n=0}^\infty c_n\,(z-z_o)^n$ with coefficients $c_n \in \mathbb{Z}$, fixed $z_o \in \mathbb{C}$, and variable $z \in \mathbb{C}$, converge *absolutely* and *uniformly* on a *disk* in $\mathbb{C}$, as opposed to converging on a more complicated region:

[1.1] Theorem: To a power series $\sum_{n=0}^\infty c_n\,(z-z_o)^n$ is attached a *radius of convergence* $0 \le R \le +\infty$, such that

$$|z-z_o| < R \implies \sum_n c_n\,(z-z_o)^n \text{ converges absolutely}$$

and

$$|z-z_o| > R \implies \sum_n c_n\,(z-z_o)^n \text{ diverges}$$

Further, for every $r < R$,

$$|z-z_o| \le r \implies \sum_n c_n\,(z-z_o)^n \text{ converges absolutely } \textit{and uniformly}$$

In particular,

$$R = \lim_n \left|\frac{c_n}{c_{n+1}}\right| \qquad \text{(if the limit exists)}$$

In general,

$$R = \liminf_n \frac{1}{\sqrt[n]{|c_n|}} = \lim_{N\to\infty} \inf_{n\ge N} \frac{1}{\sqrt[n]{|c_n|}}$$

For $R = 0$ the series converges only for $z = z_o$. For $R = +\infty$ the series converges for all z.

Proof: The conclusion in the simpler case that the indicated limit of ratios exists is reached by the *ratio test*, and the general case by a form of the *root test*, both of which are comparisons to *geometric series*.

The ratio test uses the limit

$$\lim_n \left|\frac{c_{n+1}\,(z-z_o)^{n+1}}{c_n\,(z-z_o)^n}\right| = |z-z_o| \cdot \lim_n \left|\frac{c_{n+1}}{c_n}\right|$$

if it exists. The infinite sum converges absolutely when the limit exists and is < 1:

$$|z-z_o| \cdot \lim_n \left|\frac{c_{n+1}}{c_n}\right| < 1 \implies \text{absolute convergence}$$

Oppositely, when the limit exists and is > 1, the terms do not go to 0, so the series diverges:

$$|z-z_o| \cdot \lim_n \left|\frac{c_{n+1}}{c_n}\right| > 1 \implies \text{divergence}$$

Similarly, the root test uses [1]

$$\limsup_n \sqrt[n]{|c_n\,(z-z_o)^n|} \;=\; |z-z_o|\cdot \limsup_n \sqrt[n]{|c_n|}$$

The infinite sum converges absolutely when the limsup exists and is < 1:

$$|z-z_o|\cdot \limsup_n \sqrt[n]{|c_n|} < 1 \quad\Longrightarrow\quad \text{absolute convergence}$$

Oppositely, when the limsup is > 1, the terms do not go to 0, so the series diverges:

$$|z-z_o|\cdot \limsup_n \sqrt[n]{|c_n|} > 1 \quad\Longrightarrow\quad \text{divergence}$$

The extreme cases that the radius of convergence is 0 or $+\infty$ can be treated separately.

The *uniformity* of the absolute convergence on closed disks $|z-z_o|\le r$ properly inside $|z-z_o|<R$ follows easily from convergence at every point on the circle $|z-z_o|=r$, namely, for $|z-z_o|\le r$, given $\varepsilon>0$ and N such that

$$\sum_{n\ge N} |c_n|\,r^n \;<\; \varepsilon$$

we immediately have

$$\sum_{n\ge N} |c_n\,(z-z_o)^n| \;\le\; \sum_{n\ge N} |c_n|\,r^n \;<\; \varepsilon$$

as desired. ///

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2. *Complex differentiation*

The same difference-quotient expression as in calculus of a single real variable defines the *complex* derivative of a complex-valued function $f(z)$ of a complex variable z: if the limit exists,

$$f'(z) \;=\; \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \qquad\qquad (\text{where } h \text{ is } \textit{complex})$$

The difference is that the limit is required to exist as h ranges over all small *complex* numbers. And this is much stronger than the two-dimensional real-variables requirement of differentiability.

The usual algebra shows that complex-coefficiented *polynomials* and *rational functions* are complex-differentiable.

The big surprise about complex differentiability is in Cauchy's basic theorems (smoothed-out somewhat by Goursat), which we'll come to shortly. For the moment, we restrict our attention to some important but less surprising results due to Abel about complex differentiability and *power series*.

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3. *Abel's theorem: differentiability of power series*

[3.1] Theorem: (Abel) A power series $f(z) = \sum_{n\ge 0} c_n\,(z-z_o)^n$ in one complex variable z, absolutely convergent in an open disk $|z-z_o|<r$, is *differentiable* on that disk $|z-z|<r$, and the derivative is given by the expected (absolutely convergent) series

$$f'(z) \;=\; \sum_{n\ge 0} nc_n\,z^{n-1}$$

[1] Recall that $\limsup_n a_n = \lim_N \sup_{n\ge N} a_n$.

[3.2] Corollary: Convergent power series give *smooth* (infinitely differentiable) functions. ///

[3.3] Corollary: Repeatedly differentiating,

$$f^{(k)}(z) \;=\; \sum_{n\ge 0} n(n-1)\ldots(n-k+1)\, c_n\, z^{n-k}$$

and $f^{(k)}(z_o) = k(k-1)\ldots(k-k+1)\, c_k = k!\, c_k$, so the power series coefficients of $f(z)$ are *uniquely determined* by the function f. ///

Proof: (of theorem) Without loss of generality, $z_o = 0$. Fix $0 < \rho < r$, and $|\zeta| < \rho$, $|z| < r$. Let

$$g(z) \;=\; \sum_{n\ge 0} nc_n\, z^{n-1}$$

Then

$$\frac{f(z)-f(\zeta)}{z-\zeta} - g(\zeta) \;=\; \sum_{n\ge 1} c_n \left(\frac{z^n-\zeta^n}{z-\zeta} - n\zeta^{n-1}\right)$$

For $n=1$, the expression in the parentheses is 0. For $n>1$, it is

$$z^{n-1} + z^{n-2}\zeta + z^{n-3}\zeta^2 + \ldots + z\zeta^{n-2} + \zeta^{n-1} - n\zeta^{n-1}$$

$$= (z^{n-1}-\zeta^{n-1}) + (z^{n-2}\zeta-\zeta^{n-1}) + (z^{n-3}\zeta^2-\zeta^{n-1}) + \ldots + (z^2\zeta^{n-3}-\zeta^{n-1}) + (z\zeta^{n-2}-\zeta^{n-1}) + (\zeta^{n-1}-\zeta^{n-1})$$

$$= (z-\zeta)\left[(z^{n-2}+\ldots+\zeta^{n-2}) + \zeta(z^{n-3}+\ldots+\zeta^{n-3}) + \ldots + \zeta^{n-3}(z+\zeta) + \zeta^{n-2} + 0\right]$$

$$= (z-\zeta)\sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, \zeta^k$$

For $|z|$ and $|\zeta|$ both smaller than ρ, the latter sum is dominated by

$$|z-\zeta|\,\rho^{n-2}\,\frac{n(n-1)}{2} < n^2\,|z-\zeta|\,\rho^{n-2}$$

Thus,

$$\left|\frac{f(z)-f(\zeta)}{z-\zeta} - g(\zeta)\right| \le |z-\zeta| \sum_{n\ge 2} |c_n|\, n^2\, \rho^{n-2}$$

Since $\rho < r$ the latter series converges absolutely, so the left-hand side goes to 0 as $z \to \zeta$. ///

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4. *Abel's theorem: boundary behavior*

The behavior of power series *on* the circle at the radius of convergence is much more delicate than the behavior in the interior. The power series itself may converge at *no* point on the circle, as in the example

$$\sum_{n\ge 0} nz^n \qquad \text{(converges at } \textit{no} \text{ point } |z|=1)$$

or possibly at *every* point, as in

$$\sum_{n\ge 1} \frac{z^n}{n^2} \qquad \text{(converges at } \textit{every} \text{ point } |z|=1)$$

or subtle combinations of behaviors due to not-absolute convergence: we can have convergence at all but a single boundary point, as in

$$\sum_{n\geq 1} \frac{z^n}{n} \qquad \text{(converges at every point } |z|=1 \textit{ except } z=1)$$

We can have divergence at all *roots of unity*[2] but convergence at many other boundary points, as in

$$\sum_{n\geq 1} \frac{z^{n!}}{n} \qquad \text{(diverges at roots of unity (and elsewhere) but converges at some points)}$$

[4.1] Theorem: *(Abel)* Let $f(z) = \sum_{n\geq 0} c_n\,(z-z_o)^n$ be a power series with radius of convergence $0 < R < +\infty$. Let z_1 be a point on the circle at the boundary of the radius of convergence, that is, $|z_1 - z_o| = R$. Assume that the power series f extends to a function F is holomorphic on a neighborhood of z_1. If $\sum_n c_n(z_1 - z_o)^n$ converges, then $f(z) \to F(z_1)$ when $z \to z_1$ *along a radius of the circle*. More generally, $f(z) \to F(z_1)$ when $z \to z_1$ *non-tangentially* (to the circle), that is, so that the angle $|z_1 - z|/(R - |z|)$ remains *bounded*.

[4.2] Remark: There is no general assertion of (one-sided) *differentiability*, and, indeed, any line of argument that implicitly depends on differentiability is doomed to fail.

Proof: Without loss of generality, $z_o = 0$, $R = 1$, $z_1 = 1$, and $\sum_n c_n = 0$ (the last by adjusting c_0). Let $s_n = c_0 + \ldots + c_n$ and $f_n(z) = \sum_{i=0}^n c_n\, z^n$ be the partial sums. The *summation by parts* identity is

$$f_n(z) \;=\; c_o + c_1 z + \ldots + c_n z^n \;=\; s_o + (s_1 - s_o)z + (s_2 - s_1)z^2 + \ldots + (s_n - s_{n-1})z^n$$

$$=\; s_o(1-z) + s_1(z - z^2) + \ldots + s_{n-1}(z^{n-1} - z^n) + s_n z^n \;=\; (1-z)\Big(s_o + s_1 z + s_2 z^2 + \ldots + s_{n-1} z^{n-1}\Big) + s_n z^n$$

Since $s_n \to 0$, for each fixed z with $|z| < 1$, $s_n z^n \to 0$, and

$$f(z) \;=\; \lim_n f_n(z) \;=\; (1-z)\sum_{n=0}^{\infty} s_n z^n \qquad \text{(for every } |z| < 1)$$

Given $\varepsilon > 0$, let N be large enough so that $|s_n| < \varepsilon$ for $n \geq N$, so the tail of the sum beyond N is dominated:

$$\Big|\sum_{n=N}^{\infty} s_n z^n\Big| \;\leq\; \sum_{n=N}^{\infty} \varepsilon |z|^n \;=\; \frac{\varepsilon |z|^N}{1-|z|} \;\langle\; \frac{\varepsilon}{1-|z|}$$

Using an angle restriction $|1-z|/(1-|z|) \leq C < +\infty$ (when z lies on a radius of the circle, $C = 1$),

$$|f(z)| \;\leq\; |1-z|\Big|\sum_{n=0}^{N-1} s_n z^n\Big| \;+\; |1-z|\sum_{n=N}^{\infty} \varepsilon |z|^n \;<\; |1-z|\Big|\sum_{n=0}^{N-1} s_n z^n\Big| \;+\; C\cdot\varepsilon$$

Taking $|1-z|$ sufficiently small makes the first term smaller than ε, so, for z sufficiently close to 1, within the angle restriction,

$$|f(z)| \;<\; \varepsilon + K\varepsilon \qquad \text{(for all } \varepsilon > 0)$$

[2] A complex *root of unity* is $z \in \mathbb{C}$ such that $z^N = 1$ for some positive integer N. After a little study of the exponential function, we will see that these are *dense* in the unit circle.

This holds for all $\varepsilon > 0$, so $f(z) = 0$. We had rearranged things so that $\lim_n s_n = 0$, so we have the desired result. ///

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5. *Examples*

A preliminary point is that any *polynomial* in z can easily be rewritten as a polynomial in $z - z_o$, and the latter is its power series expression *at* z_o.

Less trivially, many important power series are expansions of *rational* functions, that is, ratios of polynomials, using the fundamental summation of a geometric series

$$\frac{1}{1-z} \;=\; 1 + z + z^2 + z^3 + \ldots \qquad\qquad (\text{for } |z| < 1)$$

sometimes using *partial fraction expansions* to break the algebra into simpler pieces. For example, for $|z| < 1$ again,

$$\frac{1}{(z-1)(z-2)} \;=\; \frac{-1}{z-1} + \frac{1}{z-2} \;=\; \frac{1}{1-z} + \frac{-\frac{1}{2}}{1-\frac{z}{2}} \;=\; \Big(1 + z + z^2 + \ldots\Big) - \tfrac{1}{2}\Big(1 + \frac{z}{2} + \Big(\frac{z}{2}\Big)^2 + \ldots\Big)$$

$$= \tfrac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots$$

For that matter, later thinking in terms of *residues* will give a more efficient mnemonic for determination of the coefficients in partial fraction expansions.

Term-wise differentiation produces some interesting identities, with or without thinking about *complex* differentiation as opposed to *real*. For example, a less-familiar power series may be discovered to be an elementary function:

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + \ldots \;=\; \frac{d}{dz}\Big(1 + z + z^2 + z^3 + \ldots\Big) \;=\; \frac{d}{dz}\frac{1}{1-z} \;=\; \frac{1}{(1-z)^2}$$

Since Abel's theorem justifies differentiation of power series term-wise, it also justifies term-wise integration inside the radius of convergence: first just thinking in terms of real-variable integration rather than *path integrals*,

$$\int_0^z \sum_{n\geq 0} c_n\, w^n\; dw \;=\; \sum_{n\geq 0} c_n \int_0^z w^n\; dw \;=\; \sum_{n\geq 0} \frac{c_n}{n+1}\, z^{n+1}$$

For example,

$$\arctan z \;=\; \int_0^z \frac{dw}{1+w^2} \;=\; \int_0^z \sum_{n\geq 0} (-1)^n\, w^{2n}\; dw \;=\; \sum_{n\geq 0} (-1)^n \frac{z^{2n+1}}{2n+1} \;=\; z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$$

The series at $z = 1$ does converge, conditionally, so Abel's theorem on boundary values gives a genuine proof of Leibniz' identity

$$\frac{\pi}{4} \;=\; \arctan 1 \;=\; 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

[iou] more examples
