1. Convergence of power series

The point is that power series \( \sum_{n=0}^{\infty} c_n (z-z_o)^n \) with coefficients \( c_n \in \mathbb{Z} \), fixed \( z_o \in \mathbb{C} \), and variable \( z \in \mathbb{C} \), converge absolutely and uniformly on a disk in \( \mathbb{C} \), as opposed to converging on a more complicated region:

[1.1] Theorem: To a power series \( \sum_{n=0}^{\infty} c_n (z-z_o)^n \) is attached a radius of convergence \( 0 \leq R \leq +\infty \), such that

\[
|z-z_o| < R \implies \sum_n c_n (z-z_o)^n \text{ converges absolutely}
\]

and

\[
|z-z_o| > R \implies \sum_n c_n (z-z_o)^n \text{ diverges}
\]

Further, for every \( r < R \),

\[
|z-z_o| \leq r \implies \sum_n c_n (z-z_o)^n \text{ converges absolutely and uniformly}
\]

In particular,

\[
R = \lim_n \left| \frac{c_n}{c_{n+1}} \right| \quad \text{(if the limit exists)}
\]

In general,

\[
R = \liminf_n \frac{1}{\sqrt[n]{|c_n|}} = \lim_{N \to \infty} \inf_{n \geq N} \frac{1}{\sqrt[n]{|c_n|}}
\]

For \( R = 0 \) the series converges only for \( z = z_o \). For \( R = +\infty \) the series converges for all \( z \).

**Proof:** The conclusion in the simpler case that the indicated limit of ratios exists is reached by the ratio test, and the general case by a form of the root test, both of which are comparisons to geometric series.

The ratio test uses the limit

\[
\lim_n \left| \frac{c_{n+1} (z-z_o)^{n+1}}{c_n (z-z_o)^n} \right| = |z-z_o| \cdot \lim_n \left| \frac{c_{n+1}}{c_n} \right|
\]

if it exists. The infinite sum converges absolutely when the limit exists and is \( < 1 \):

\[
|z-z_o| \cdot \lim_n \left| \frac{c_{n+1}}{c_n} \right| < 1 \implies \text{absolute convergence}
\]

Oppositely, when the limit exists and is \( > 1 \), the terms do not go to 0, so the series diverges:

\[
|z-z_o| \cdot \lim_n \left| \frac{c_{n+1}}{c_n} \right| > 1 \implies \text{divergence}
\]
Similarly, the root test uses the formula
\[ \limsup_n \sqrt[n]{|c_n (z - z_o)^n|} = |z - z_o| \limsup_n \sqrt[n]{|c_n|} \]
The infinite sum converges absolutely when the limsup exists and is < 1:
\[ |z - z_o| \limsup_n \sqrt[n]{|c_n|} < 1 \implies \text{absolute convergence} \]
Oppositely, when the limsup is > 1, the terms do not go to 0, so the series diverges:
\[ |z - z_o| \limsup_n \sqrt[n]{|c_n|} > 1 \implies \text{divergence} \]
The extreme cases that the radius of convergence is 0 or \( +\infty \) can be treated separately.

The uniformity of the absolute convergence on closed disks \( |z - z_o| \leq r \) properly inside \( |z - z_o| < R \) follows easily from convergence at every point on the circle \( |z - z_o| = r \), namely, for \( |z - z_o| \leq r \), given \( \varepsilon > 0 \) and \( N \) such that
\[ \sum_{n \geq N} |c_n| r^n < \varepsilon \]
we immediately have
\[ \sum_{n \geq N} |c_n (z - z_o)^n| \leq \sum_{n \geq N} |c_n| r^n < \varepsilon \]
as desired. ///

2. Complex differentiation

The same difference-quotient expression as in calculus of a single real variable defines the complex derivative of a complex-valued function \( f(z) \) of a complex variable \( z \): if the limit exists,
\[ f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} \quad \text{(where \( h \) is complex)} \]
The difference is that the limit is required to exist as \( h \) ranges over all small complex numbers. And this is much stronger than the two-dimensional real-variables requirement of differentiability.

The usual algebra shows that complex-coefficiented polynomials and rational functions are complex-differentiable.

The big surprise about complex differentiability is in Cauchy’s basic theorems (smoothed-out somewhat by Goursat), which we’ll come to shortly. For the moment, we restrict our attention to some important but less surprising results due to Abel about complex differentiability and power series.

3. Abel’s theorem: differentiability of power series

[3.1] Theorem: (Abel) A power series \( f(z) = \sum_{n \geq 0} c_n (z - z_o)^n \) in one complex variable \( z \), absolutely convergent in an open disk \( |z - z_o| < r \), is differentiable on that disk \( |z - z| < r \), and the derivative is given by the expected (absolutely convergent) series
\[ f'(z) = \sum_{n \geq 0} nc_n (z - z_o)^{n-1} \]

[1] Recall that \( \limsup_n a_n = \limsup N \sup_{n \geq N} a_n \).
[3.2] **Corollary**: Convergent power series give smooth (infinitely differentiable) functions. ///

[3.3] **Corollary**: Repeatedly differentiating,

\[ f^{(k)}(z) = \sum_{n \geq 0} n(n - 1) \ldots (n - k + 1) c_n (z - z_o)^{n-k} \]

and \( f^{(k)}(z_o) = k(k-1) \ldots (k-k+1) c_k = k! c_k \), so the power series coefficients of \( f(z) \) are uniquely determined by the function \( f \). ///

**Proof**: (of theorem) Without loss of generality, \( z_o = 0 \). Fix \( 0 < \rho < r \), and \( |\zeta| < \rho \), \( |z| < r \). Let \( g(z) = \sum_{n \geq 0} n c_n z^{n-1} \)

Then \( \frac{f(z) - f(\zeta)}{z - \zeta} - g(\zeta) = \sum_{n \geq 1} c_n \left( \frac{z^n - \zeta^n}{z - \zeta} - n\zeta^{n-1} \right) \)

For \( n = 1 \), the expression in the parentheses is 0. For \( n > 1 \), it is

\[
\begin{align*}
(z^n &- \zeta^n) + (z^{n-2} \zeta - \zeta^{n-1}) + (z^{n-3} \zeta^2 - \zeta^{n-1}) + \ldots + (z^2 \zeta^{n-3} - \zeta^{n-1}) + (z \zeta^{n-2} - \zeta^{n-1}) + (\zeta^{n-1} - \zeta^{n-1}) \\
&= (z - \zeta) \left[ (z^{n-2} + \ldots + \zeta^{n-2}) + \zeta(z^{n-3} + \ldots + \zeta^{n-3}) + \ldots + \zeta^{n-3}(z + \zeta) + \zeta^{n-2} + 0 \right] \\
&= (z - \zeta) \sum_{k=0}^{n-2} (k + 1) z^{n-2-k} \zeta^k
\end{align*}
\]

For \( |z| \) and \( |\zeta| \) both smaller than \( \rho \), the latter sum is dominated by

\[
|z - \zeta| \rho^{n-2} \frac{n(n-1)}{2} < n^2 |z - \zeta| \rho^{n-2}
\]

Thus,

\[
\left| \frac{f(z) - f(\zeta)}{z - \zeta} - g(\zeta) \right| \leq |z - \zeta| \sum_{n \geq 2} |c_n| n^2 \rho^{n-2}
\]

Since \( \rho < r \) the latter series converges absolutely, so the left-hand side goes to 0 as \( z \to \zeta \). ///

4. **Abel’s theorem: boundary behavior**

The behavior of power series on the circle at the radius of convergence is much more delicate than the behavior in the interior. The power series itself may converge at no point on the circle, as in the example

\[
\sum_{n \geq 0} n z^n \quad \text{(converges at no point } |z| = 1) \]

or possibly at every point, as in

\[
\sum_{n \geq 1} \frac{z^n}{n^2} \quad \text{(converges at every point } |z| = 1) \]
or subtle combinations of behaviors due to not-absolute convergence: we can have convergence at all but a single boundary point, as in

$$ \sum_{n \geq 1} \frac{z^n}{n} $$

(converges at every point $|z| = 1$ except $z = 1$)

We can have divergence at all roots of unity\(^2\) but convergence at many other boundary points, as in

$$ \sum_{n \geq 1} \frac{z^n}{n} $$

(diverges at roots of unity (and elsewhere) but converges at some points)

[4.1] Theorem: (Abel) Let $f(z) = \sum_{n \geq 0} c_n (z - z_o)^n$ be a power series with radius of convergence $0 < R < +\infty$. Let $z_1$ be a point on the circle at the boundary of the radius of convergence, that is, $|z_1 - z_o| = R$. Assume that the power series $f$ extends to a function $F$ is holomorphic on a neighborhood of $z_1$. If $\sum_{n} c_n (z_1 - z_o)^n$ converges, then $f(z) \to F(z_1)$ when $z \to z_1$ along a radius of the circle. More generally, $f(z) \to F(z_1)$ when $z \to z_1$ non-tangentially (to the circle), that is, so that the angle $|z_1 - z|/(R - |z|)$ remains bounded.

[4.2] Remark: There is no general assertion of (one-sided) differentiability, and, indeed, any line of argument that implicitly depends on differentiability is doomed to fail.

**Proof:** Without loss of generality, $z_o = 0$, $R = 1$, $z_1 = 1$, and $\sum_{n} c_n = 0$ (the last by adjusting $c_0$). Let $s_n = c_0 + \ldots + c_n$ and $f_n(z) = \sum_{i=0}^{n} c_n z^n$ be the partial sums. The summation by parts identity is

$$ f_n(z) = c_o + c_1 z + \ldots + c_n z^n = s_o + (s_1 - s_o)z + (s_2 - s_1)z^2 + \ldots + (s_n - s_{n-1})z^n $$

$$ = s_o(1-z) + s_1(z - z^2) + \ldots + s_{n-1}(z^{n-1} - z^n) + s_n z^n = (1-z) \left( s_o + s_1 z + s_2 z^2 + \ldots + s_{n-1} z^{n-1} \right) + s_n z^n $$

Since $s_n \to 0$, for each fixed $z$ with $|z| < 1$, $s_n z^n \to 0$, and

$$ f(z) = \lim_{n} f_n(z) = (1-z) \sum_{n=0}^{\infty} s_n z^n \quad \text{(for every } |z| < 1) $$

Given $\varepsilon > 0$, let $N$ be large enough so that $|s_n| < \varepsilon$ for $n \geq N$, so the tail of the sum beyond $N$ is dominated:

$$ \left| \sum_{n=N}^{\infty} s_n z^n \right| \leq \sum_{n=N}^{\infty} |s_n| |z|^n = \frac{\varepsilon |z|^N}{1 - |z|} \left\{ \frac{\varepsilon}{1 - |z|} \right\} $$

Using an angle restriction $|1-z|/(1-|z|) \leq C < +\infty$ (when $z$ lies on a radius of the circle, $C = 1$),

$$ |f(z)| \leq |1-z| \sum_{n=0}^{N-1} s_n z^n + |1-z| \sum_{n=N}^{\infty} |s_n| |z|^n < |1-z| \sum_{n=0}^{N-1} s_n z^n + C \cdot \varepsilon $$

Taking $|1-z|$ sufficiently small makes the first term smaller than $\varepsilon$, so, for $z$ sufficiently close to 1, within the angle restriction,

$$ |f(z)| < \varepsilon + K\varepsilon \quad \text{ (for all } \varepsilon > 0) $$

\(^2\) A complex root of unity is $z \in \mathbb{C}$ such that $z^N = 1$ for some positive integer $N$. After a little study of the exponential function, we will see that these are dense in the unit circle.
This holds for all $\varepsilon > 0$, so $f(z) = 0$. We had rearranged things so that $\lim_n s_n = 0$, so we have the desired result.

///

5. Examples

A preliminary point is that any polynomial in $z$ can easily be rewritten as a polynomial in $z - z_0$, and the latter is its power series expression at $z_0$.

Less trivially, many important power series are expansions of rational functions, that is, ratios of polynomials, using the fundamental summation of a geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \quad \text{(for } |z| < 1)$$

sometimes using partial fraction expansions to break the algebra into simpler pieces. For example, for $|z| < 1$ again,

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} + \frac{1}{1-z^2} = \left(1 + z + z^2 + \ldots\right) - \frac{1}{2} \left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \ldots\right)$$

$$= \frac{1}{2} + \frac{3}{4} z + \frac{7}{8} z^2 + \frac{15}{16} z^3 + \ldots$$

For that matter, later thinking in terms of residues will give a more efficient mnemonic for determination of the coefficients in partial fraction expansions.

Term-wise differentiation produces some interesting identities, with or without thinking about complex differentiation as opposed to real. For example, a less-familiar power series may be discovered to be an elementary function:

$$1 + 2z + 3z^2 + 4z^3 + 5z^4 + \ldots = \frac{d}{dz} \left(1 + z + z^2 + z^3 + \ldots\right) = \frac{d}{dz} \frac{1}{1-z} = \frac{1}{(1-z)^2}$$

Since Abel’s theorem justifies differentiation of power series term-wise, it also justifies term-wise integration inside the radius of convergence: first just thinking in terms of real-variable integration rather than path integrals,

$$\sum_{n \geq 0} c_n w^n \, dw = \sum_{n \geq 0} c_n \int_0^z w^n \, dw = \sum_{n \geq 0} \frac{c_n}{n+1} z^{n+1}$$

For example,

$$\arctan z = \int_0^z \frac{dw}{1 + w^2} = \int_0^z \sum_{n \geq 0} (-1)^n w^{2n} \, dw = \sum_{n \geq 0} (-1)^n \frac{z^{2n+1}}{2n+1} = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots$$

The series at $z = 1$ does converge, conditionally, so Abel’s theorem on boundary values gives a genuine proof of Leibniz’ identity

$$\pi = 4 \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

[jou] more examples