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03. The exponential function, sine, cosine

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[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2020-21/03_exp_sin_cos.pdf]

We attempt to solve the differential equation (with initial condition)

$$u' = u \quad u(0) = 1$$

in terms of a power series

$$u(x) = \sum_{n=0}^{\infty} c_n x^n$$

From Abel's theorem, assuming convergence,

$$u'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \quad u(0) = c_0$$

and the coefficients are uniquely determined by the values of the function. This implies a sequence of equalities

$$c_0 = 1, \quad 1 \cdot c_1 = c_0, \quad 2 \cdot c_2 = c_1, \quad 3 \cdot c_3 = c_2, \quad \dots$$

from which^[1] $c_n = 1/n!$ and

$$u(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

This outcome was under an assumption that the series so-obtained is convergent, and, by the ratio test, it is indeed (absolutely) convergent for all real x , and, similarly, for all *complex* x . Write

$$e^z = \sum_{n \geq 0} \frac{z^n}{n!}$$

Note that it converges *uniformly* absolutely for z in compact subsets of the complex plane.

[0.1] **Proposition:** For complex z, w ,

$$e^{z+w} = e^z \cdot e^w$$

The *complex conjugate* $\overline{e^z}$ of e^z is $e^{\bar{z}}$.

Proof: The first assertion is a corollary of the binomial theorem, as follows. With absolute convergence justifying rearrangements:

$$e^{z+w} = \sum_{n \geq 0} \frac{(z+w)^n}{n!} = \sum_{n,i} \binom{n}{i} z^{n-i} w^i / n!$$

with usual binomial coefficients

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

The $n!$'s cancel, leaving

$$e^{z+w} = \sum_{n,i} \frac{1}{i!(n-i)!} z^{n-i} w^i = e^z \cdot e^w$$

[1] Recall that the *factorials* $n!$ for $n = 1, 2, \dots$ are $n! = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1$ and $0! = 1$.

The second assertion is a consequence of the fact that the power series has real coefficients, and that complex conjugation is a *continuous* map of the complex numbers to themselves. Thus, since the partial sums of e^z conjugate as indicated, the limit does as well. ///

[0.2] Corollary: $(e^z)^n = e^{nz}$ for $n \in \mathbb{Z}$.

Proof: For positive integers n , by induction

$$(e^z)^{n+1} = (e^z)^n \cdot e^z = e^{nz} \cdot e^z = e^{nz+z} = e^{(n+1)z}$$

Next, since

$$1 = e^0 = e^{z+(-z)} = e^z \cdot e^{-z}$$

we have $(e^z)^{-1} = e^{-z}$, and a similar induction gives $(e^z)^{-n} = e^{-nz}$ for negative integers $-n$. ///

[0.3] Corollary: $|e^{ix}| = 1$ for real x .

Proof: Simply

$$|e^{ix}|^2 = e^{ix} \overline{e^{ix}} = e^{ix} e^{-ix} = e^0 = 1$$

using $x \in \mathbb{R}$. ///

We grant that the trigonometric functions $\sin x$ and $\cos x$ satisfy the differential equation $u'' = -u$ with boundary conditions

$$\sin' 0 = \cos 0 = 1 \quad \cos' 0 = \sin 0 = 0$$

Again solving for *power series* solutions, first assuming convergence, we obtain

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Then

$$e^{iz} = \cos z + i \sin z$$

for all real or complex z .

[0.4] Corollary: For real or complex z

$$\cos^2 z + \sin^2 z = 1$$

Proof: We have

$$1 = e^{iz} e^{-iz} = (\cos z + i \sin z)(\cos z - i \sin z) = \cos^2 z + \sin^2 z$$

for any complex z . ///

[0.5] Lemma: The function $\cos x$ defined by the power series has *least positive zero* between $\frac{4}{3}$ and $\frac{5}{3}$.

Proof: From elementary estimates, noting that everything is *real*, for $0 \leq x \leq \frac{4}{3}$ the series for $\cos x$ is *alternating decreasing* after the first two terms, so

$$\cos x \geq 1 - \frac{x^2}{2} \geq 1 - \frac{\left(\frac{4}{3}\right)^2}{2!} = 1 - \frac{8}{9} > 0$$

while

$$\cos \frac{5}{3} \leq 1 - \frac{\left(\frac{5}{3}\right)^2}{2!} + \frac{\left(\frac{5}{3}\right)^4}{4!} = 1 - \frac{25}{18} + \frac{625}{81 \cdot 24} = -\frac{131}{1944} < 0$$

By the intermediate value theorem, there is a 0 between. ///

Declaring π to be defined by saying that the first positive real 0 of $\cos x$ is $\pi/2$,

[0.6] Corollary:

$$e^{z+2\pi i} = e^z \quad e^{\pi i} = -1 \quad e^{\pi i/2} = i$$