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05. The Gamma function $\Gamma(s)$ *[draft]*

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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1. *Euler's integral for* $\Gamma(s)$

The Gamma function $\Gamma(s)$ can be defined by Euler's integral

$$\Gamma(s) \; = \; \int_0^\infty t^s \, e^{-t} \, \frac{dt}{t} \qquad \text{(absolute convergence for } \mathrm{Re}(s) > 0\text{)}$$

Integration by parts proves the *functional equation*

$$\Gamma(s+1) \; = \; s \cdot \Gamma(s) \qquad \text{(for } \mathrm{Re}(s) > 0\text{)}$$

For $0 < s \in \mathbb{Z}$, the functional equation and induction show the connection to *factorials*:

$$\Gamma(n) \; = \; (n-1)! \qquad \text{(for } n = 1, 2, \ldots\text{)}$$

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2. *Holomorphy of integrals*

It is not surprising that $\Gamma(s)$ is holomorphic in the region of absolute convergence $\mathrm{Re}(s) > 0$. This can be proven by checking complex differentiability of truncated integrals, and invoking the holomorphy of uniform-on-compact limits of holomorphic functions. Alternatively, but essentially equivalently in terms of fundamental invocation of Cauchy's theorem and corollaries, holomorphy can be proven via Morera's theorem, invoking Fubini-Tonelli to justify interchange of integrals. Both approaches are typical for proving holomorphy of integrals with a parameter, when the integrands are holomorphic functions of the parameter. In this section we recall some broadly applicable ideas.

[2.1] Claim: Let $F(t,z)$ be a function of $t \in [a,b] \subset \mathbb{R}$ and $z \in \Omega \subset \mathbb{C}$ with non-empty open Ω, continuous as a function of the two variables, and holomorphic in z for each fixed t. Then

$$f(z) \; = \; \int_a^b F(t,z) \, dt$$

is holomorphic for $z \in \Omega$. Further, the complex derivative is

$$f'(z) \; = \; \int_a^b \frac{\partial F}{\partial z}(t,z) \, dt$$

where $\frac{\partial F}{\partial z}$ is the complex derivative in the second argument of F. That is, the operator of complex differentiation passes inside the integral.

[2.2] Remark: Without compactness or similar hypothesis on the behavior in the integration variable, the conclusion can easily fail, and in non-pathological ways, for example,

$$f(z) \;=\; \int_{-\infty}^{\infty} \frac{e^{itz}\,dt}{1+t^2}$$

is *not* holomorphic in z. The integral does not converge at all for $z \notin \mathbb{R}$.

Proof: First, we claim that F_z and F_{zz}, the first and second complex derivatives of F in its second argument, are continuous as functions of their two arguments. From Cauchy's integral formulas, for each fixed $t \in [a,b]$, for any simple closed path γ around z_o, inside Ω, for any z inside γ,

$$F(t,z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{F(t,w)\,dw}{w-z} \qquad \text{and} \qquad F_z(t,z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{F(t,w)\,dw}{(w-z)^2}$$

and similarly for F_{zz}. Given $z_o \in \Omega$, let B_{2r}, B_r be open balls of radius $2r, r$ centered at z_o and so that the closure of B_{2r} fits inside Ω. We may as well let γ be the boundary of B_{2r}, traversed in a positive direction. Let C_r be the closure of B_r. The continuity of F on the compact set $[a,b] \times C_r$ implies *uniform* continuity on that set, and on $[a,b] \times \gamma$.

Using that joint continuity, given $\varepsilon > 0$, take $\delta > 0$ such that $|z - z_o| < \delta$ implies $|(w-z)^{-2} - (w-z_o)^{-2}| < \varepsilon$ for all $w \in \gamma$. Let M be the maximum of the continuous function $F(t,w)$ on the compact $[a,b] \times C_r$. By the trivial estimate on the Cauchy formula integral,

$$\Big|\int_\gamma \frac{F(t,w)\,dw}{(w-z)^2} \;-\; \int_\gamma \frac{F(t,w)\,dw}{(w-z_o)^2}\Big| \;\le\; 2\pi 2r \cdot M \cdot \max_{w\in\gamma}\Big|\frac{1}{(w-z)^2} - \frac{1}{(w-z_o)^2}\Big| \;\le\; 2\pi 2r \cdot M \cdot \varepsilon$$

This gives the continuity of $F_z(t,z)$. A nearly identical argument gives that of $F_{zz}(t,z)$.

By the complex differentiability in z, for fixed z_o, for every $t \in [a,b]$ and $z \in C_r$,

$$F(t,z) \;=\; F(t,z_o) + (z-z_o)F_z(t,z_o) + R(t,z)$$

where the remainder $R(t,z)$ satisfies a uniform estimate of the form

$$|R(t,z)| \;\le\; B \cdot |z-z_o|^2 \qquad\qquad (\text{for all } (t,z) \in [a,b] \times C_r)$$

Thus,

$$\Big|\int_{[a,b]} F(t,z)\,dt - \int_{[a,b]} F(t,z_o)\,dt\Big| \;\le\; \int_{[a,b]} |F(t,z) - F(t,z_o)|\,dt \;\le\; \int_{[a,b]} B\cdot|z-z_o|^2\,dt \;=\; |b-a|\cdot B\cdot|z-z_o|^2$$

Thus,

$$\Big|\int_{[a,b]} \frac{F(t,z)-F(t,z_o)}{z-z_o} - F_z(t,z_o)\,dt\Big| \;\le\; |b-a|\cdot B\cdot|z-z_o| \;\longrightarrow\; 0 \qquad\qquad (\text{as } z \to z_o)$$

This proves the complex differentiability of the integral in t, and identifies the derivative as the corresponding integral of F_z. That is, the complex differentiation in z passes inside the integral, as hoped. ///

As an example of limits of compact integrals that are still holomorphic:

[2.3] Claim: Let $F(t,z)$ be continuous in $t \in (0,\infty)$ and complex differentiable in z in non-empty open Ω. Assume that

$$f(z) \;=\; \int_0^\infty F(t,z)\;dt$$

is absolutely convergent for all $z \in \Omega$. Assume that, for every compact $K \subset \Omega$,

$$\lim_{a\to 0^+,\, b\to+\infty} \int_a^b F(t,z)\;dt \;=\; \int_0^\infty F(t,z)\;dt \qquad\qquad \text{(uniformly for } z \in K)$$

That is, given compact $K \subset \Omega$, given $\varepsilon > 0$, there exist a_o, b_o such that, for all $z \in K$, and for all $0 < a, a' \le a_o$ and for all $b, b' \ge b_o$,

$$\Big| \int_a^b F(t,z)\;dt - \int_{a'}^{b'} F(t,z)\;dt \Big| \;<\; \varepsilon$$

Then $\int_0^\infty F(t,z)\;dt$ is holomorphic in $z \in \Omega$, and its complex derivative is $\int_0^\infty F_z(t,z)\;dt$.

Proof: The previous claim shows that all the truncated integrals $f_{a,b}(z) = \int_a^b F(t,z)\;dt$ are holomorphic. The hypothesis is exactly that the functions $f_{a,b}$ converge pointwise, uniformly on compacts, to the infinite integral. A uniform-on-compacts pointwise limit of holomorphic functions is holomorphic. ///

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3. *Holomorphy of* $\Gamma(s)$ *in* $\mathrm{Re}(s) > 0$

The general claims of the previous section give

[3.1] Claim: The integral $\displaystyle\int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}$ is a holomorphic function of complex s for $\mathrm{Re}(s) > 0$.

Proof: The cases that $0 < \mathrm{Re}(s) \le 1$ and $1 \le \mathrm{Re}(s)$ are somewhat different, due to the corresponding behaviors of t^s near 0 and near $+\infty$.

For $\mathrm{Re}(s) \ge 1$, with the logarithm that is real-valued on $(0,+\infty)$, for $0 < b < b'$,

$$|t^{s-1}| \;=\; |e^{(s-1)\log t}| \;=\; e^{\mathrm{Re}\,((s-1)\log t)} \;=\; e^{(\mathrm{Re}\,(s)-1)\log t} \;=\; t^{\mathrm{Re}\,(s)-1}$$

Then

$$\Big|\int_0^b e^{-t}\,t^s\,\frac{dt}{t} \;-\; \int_0^{b'} e^{-t}\,t^s\,\frac{dt}{t}\Big| \;\le\; \int_b^{b'} e^{-t}\,t^{\mathrm{Re}\,(s)}\,\frac{dt}{t} \;\le\; \int_b^\infty e^{-t}\,t^{\mathrm{Re}\,(s)}\,\frac{dt}{t}$$

$$\le\; \int_b^\infty e^{-t/2}\,e^{-t/2}t^{\mathrm{Re}\,(s)}\,\frac{dt}{t} \;=\; \int_b^\infty e^{-t/2}\,dt \times \sup_{t\ge b} e^{-t/2}t^{\mathrm{Re}\,(s)-1} \;=\; b^{-t/2} \times \sup_{t\ge b} e^{-t/2}t^{\mathrm{Re}\,(s)}$$

Given compact K, there is σ_1 such that $s \in K$ implies $\mathrm{Re}\,(s) \le \sigma_1$. The sup is finite, so we have exponential decay in b, giving the uniform estimate

$$\Big|\int_0^b e^{-t}\,t^s\,\frac{dt}{t} \;-\; \int_0^{b'} e^{-t}\,t^s\,\frac{dt}{t}\Big| \;\le\; b^{-t/2} \times \sup_{t\ge b} e^{-t/2}t^{\sigma_1}$$

for $s \in K$. For $0 < \mathrm{Re}\,(s) \le 1$, the convergence of $\int_0^1 e^{-t}\,t^s\,dt/t$ implies that

$$\lim_{a,a'\to 0} \int_a^{a'} e^{-t}\,t^s\,\frac{dt}{t} \;\longrightarrow\; 0$$

Then similar estimates give the uniform-on-compacts convergence. ///

4. *Meromorphic continuation of* $\Gamma(s)$ *to* $\mathbb{C}$

From the functional equation, we get a meromorphic continuation of $\Gamma(s)$ to the entire complex plane, except for poles at non-positive integers $-n$. The poles are *simple*, with residue $(-1)^n/n!$ at $-n$.

[... iou ...]

5. $\int_0^\infty e^{-tz} t^s \frac{dt}{t} = z^{-s}\Gamma(s)$

The identity

$$\int_0^\infty t^s\, e^{-ty}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{y^s} \qquad\qquad (\text{for } y > 0 \text{ and } \mathrm{Re}(s) > 0)$$

for $y > 0$ first follows for $\mathrm{Re}(s) > 0$ by replacing t by t/y in the integral. Then

$$\int_0^\infty t^s\, e^{-tz}\, \frac{dt}{t} \;=\; \frac{\Gamma(s)}{z^s} \qquad\qquad (\text{for } \mathrm{Re}(z) > 0 \text{ and } \mathrm{Re}(s) > 0)$$

by complex analysis, since both sides are holomorphic in s and agree on the positive reals.

The latter identity allows non-obvious evaluation of a Fourier transform. Namely, let

$$f(x) \;=\; \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

For $\mathrm{Re}(\alpha) > -1$ this function is locally integrable at 0, and in any case is of rapid decay at infinity. We can compute its Fourier transform:

$$\int_{\mathbb{R}} e^{-2\pi i\xi x}\, f(x)\, dx \;=\; \int_0^\infty e^{-2\pi i \xi x}\, x^{\alpha+1}\, e^{-x}\, \frac{dx}{x} \;=\; \int_0^\infty x^{\alpha+1}\, e^{-x(1+2\pi i\xi)}\, \frac{dx}{x} \;=\; \frac{\Gamma(\alpha+1)}{(1+2\pi i\xi)^{\alpha+1}}$$

Further, Fourier inversion gives the non-obvious

$$\int_{\mathbb{R}} e^{2\pi i \xi x}\, \frac{1}{(1+2\pi i\xi)^{\alpha+1}}\, d\xi \;=\; \frac{1}{\Gamma(\alpha+1)} \cdot \begin{cases} x^\alpha \cdot e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x < 0) \end{cases}$$

For $\alpha \in \mathbb{Z}$, the same conclusion can be reached by evaluation by residues.

6. *Euler's Beta integral in terms of* Γ

[6.1] Claim: Euler's beta integral

$$B(a,b) \;=\; \int_0^1 x^{a-1}\, (1-x)^{b-1}\, dx$$

is expressible in terms of Γ as

$$B(a,b) \;=\; \int_0^1 x^{a-1}\, (1-x)^{b-1}\, dx \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

Proof: Replacing x by $\frac{t}{t+1} = 1 - \frac{1}{t+1}$ in the integral gives

$$\int_0^1 x^{a-1}\, (1-x)^{b-1}\, dx \;=\; \int_0^\infty \Big(\frac{t}{t+1}\Big)^{a-1} \Big(1 - \frac{t}{t+1}\Big)^{b-1} \frac{dt}{(t+1)^2} \;=\; \int_0^\infty t^a \Big(\frac{1}{t+1}\Big)^{a+b} \frac{dt}{t}$$

Use the gamma identity in the form

$$\Big(\frac{1}{t+1}\Big)^s \;=\; \frac{1}{\Gamma(s)}\int_0^\infty e^{-u(t+1)}\,u^s\,\frac{du}{u}$$

to rewrite the beta integral further as

$$\frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^{a+b}\,t^a\,e^{-u(t+1)}\,\frac{du}{u}\frac{dt}{t} \;=\; \frac{1}{\Gamma(a+b)}\int_0^\infty\int_0^\infty u^b\,t^a\,e^{-u}\,e^{-t}\,\frac{dt}{t}\frac{du}{u} \;=\; \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$$

as claimed. ///

[6.2] Remark: If we add another similar factor to the Beta integral, we have Euler's integral representation for *hypergeometric functions*, namely,

$$F(\alpha,\beta,\gamma;z) \;=\; \frac{1}{B(\beta,\gamma-\beta)}\int_0^1 x^{\beta-1}\,(1-x)^{\gamma-\beta-1}\,(1-xz)^{-\alpha}\,dx$$

This F is the ${}_2F_1$ hypergeometric function, whose *series* definition is

$$F(\alpha,\beta,\gamma;z) \;=\; 1+\frac{a\,b}{c}\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!}+\ldots \;=\; \sum_{n=0}^\infty \frac{(a)_n\,(b)_n}{(c)_n}\,\frac{z^n}{n!}$$

The notation $(a)_n$ is the *Pockhammer* symbol.

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7. $\Gamma(s)\cdot\Gamma(1-s) = \pi/\sin\pi s$

To prove this, take $0<\mathrm{Re}(s)<1$ for convergence of both integrals, and compute

$$\Gamma(s)\cdot\Gamma(1-s) \;=\; \int_0^\infty\int_0^\infty u^s\,e^{-u}\cdot v^{1-s}\,e^{-v}\,\frac{du}{u}\,\frac{dv}{v} \;=\; \int_0^\infty\int_0^\infty u\,e^{-u(1+v)}\,v^{1-s}\,\frac{du}{u}\,\frac{dv}{v}$$

by replacing v by uv. Replacing u by $u/(1+v)$ (another instance of the basic *gamma identity*) and noting that $\Gamma(1)=1$ gives

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv$$

Replace the path from 0 to ∞ by the *Hankel contour* H_ε described as follows. Far to the right on the real line, start with the branch of v^{-s} given by $(e^{2\pi i}v)^{-s}=e^{-2\pi is}v^{-s}$, integrate from $+\infty$ to $\varepsilon>0$ along the real axis, clockwise around a circle of radius ε at 0, then back out to $+\infty$, now with the standard branch of v^{-s}. For $\mathrm{Re}(-s)>-1$ the integral around the little circle goes to 0 as $\varepsilon\to 0$. Thus,

$$\int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \lim_{\varepsilon\to 0}\frac{1}{1-e^{-2\pi is}}\int_{H_\varepsilon}\frac{v^{-s}}{1+v}\,dv$$

The integral of this integrand over a large circle goes to 0 as the radius goes to $+\infty$, for $\mathrm{Re}(-s)<0$. Thus, this integral is equal to the limit as $R\to+\infty$ and $\varepsilon\to 0$ of the integral

from R to ε
from ε clockwise back to ε
from ε to R
from R counterclockwise to R

This integral is $2\pi i$ times the sum of the residues inside it, namely, that at $v = -1 = e^{\pi i}$. Thus,

$$\Gamma(s) \cdot \Gamma(1-s) \;=\; \int_0^\infty \frac{v^{-s}}{1+v}\,dv \;=\; \frac{2\pi i}{1-e^{-2\pi i s}} \cdot (e^{\pi i})^{-s} \;=\; \frac{2\pi i}{e^{\pi i s} - e^{-\pi i s}} \;=\; \frac{\pi}{\sin \pi s}$$

as claimed. ///

[7.1] Corollary: $\Gamma(s) \neq 0$ for all $s \in \mathbb{C}$. ///

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8. *Duplication:* $\Gamma(s) \cdot \Gamma(s+\frac{1}{2}) \;=\; 2^{1-2s} \cdot \sqrt{\pi} \cdot \Gamma(2s)$

To prove this, from the Eulerian integral definition,

$$\Gamma(s) \cdot \Gamma(s+\tfrac{1}{2}) \;=\; \int_0^\infty e^{-t}\, t^s\, \frac{dt}{t} \cdot \int_0^\infty e^{-u}\, u^{s+\frac{1}{2}}\, \frac{du}{u}$$

Replacing t by t/u

$$\int_0^\infty \int_0^\infty e^{-(\frac{t}{u}+u)}\, t^s\, u^{\frac{1}{2}}\, \frac{du}{u}\, \frac{dt}{t}$$

In the Fourier transform identity

$$e^{-\pi\xi^2} \;=\; \int_{\mathbb{R}} e^{-2\pi i x\cdot\xi}\, e^{-\pi x^2}\, dx$$

let $\xi = \sqrt{t}/\sqrt{u}$ and replace x by $x/\sqrt{\pi}$:

$$e^{-\pi\frac{t}{u}} \;=\; \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2\pi i x\cdot\frac{\sqrt{t}}{\sqrt{u}\sqrt{\pi}}}\, e^{-x^2}\, dx$$

and replace t by t/π to obtain

$$e^{-\frac{t}{u}} \;=\; \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-2ix\cdot\frac{\sqrt{t}}{\sqrt{u}}}\, e^{-x^2}\, dx$$

Substituting the Fourier transform expression in place of $e^{-\frac{t}{u}}$ gives

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix\cdot\frac{\sqrt{t}}{\sqrt{u}}}\, e^{-x^2}\, e^{-u}\, t^s\, u^{\frac{1}{2}}\, dx\, \frac{du}{u}\, \frac{dt}{t}$$

Replace x by $x\sqrt{u}$, and then u by $u/(x^2+1)$:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix\cdot\sqrt{t}}\, e^{-u(x^2+1)}\, t^s\, u\, dx\, \frac{du}{u}\, \frac{dt}{t} \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^\infty \int_{\mathbb{R}} e^{-2ix\cdot\sqrt{t}}\, \frac{1}{x^2+1} e^{-u}\, t^s\, u\, dx\, \frac{du}{u}\, \frac{dt}{t}$$

$$=\; \frac{1}{\sqrt{\pi}}\Gamma(1) \int_0^\infty \int_{\mathbb{R}} e^{-2ix\cdot\sqrt{t}}\, \frac{1}{x^2+1}\, t^s\, dx\, \frac{dt}{t} \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}} e^{-2ix\cdot\sqrt{t}}\, \frac{1}{x^2+1}\, t^s\, dx\, \frac{dt}{t}$$

The inner integral over x can be evaluated by residues: it captures the negative of the residue of $x \to e^{-2ix\sqrt{t}}/(x^2+1)$ in the *lower* half-plane, giving

$$\int_{\mathbb{R}} e^{-2ix\cdot\sqrt{t}}\, \frac{1}{x^2+1}\, dx \;=\; -2\pi i \cdot e^{-2i(-i)\sqrt{t}} \cdot \frac{1}{(-i)-i} \;=\; \pi\, e^{-2\sqrt{t}}$$

Summarizing, and then replacing t by t^2 and t by $t/2$:

$$\Gamma(s)\cdot\Gamma(s+\tfrac{1}{2}) \;=\; \sqrt{\pi} \int_0^\infty e^{-2\sqrt{t}}\, t^s\, \frac{dt}{t} \;=\; 2\sqrt{\pi} \int_0^\infty e^{-2t}\, t^{2s}\, \frac{dt}{t} \;=\; 2^{1-2s}\, \sqrt{\pi} \int_0^\infty e^{-t}\, t^{2s}\, \frac{dt}{t} \;=\; \sqrt{\pi}\, 2^{1-2s}\Gamma(2s)$$

as claimed. ///
