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07. Compactification: Riemann sphere, projective space

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1. The Riemann sphere

One traditional *one-point compactification* of $\mathbb{C}$ can be picturesquely extrinsically described via the *stereographic projection* map from the unit sphere $S^2 \subset \mathbb{R}^3$, with the point $(x, y, z) = (0, 0, 1)$ removed, to the x, y-plane. The same device applies to $\mathbb{R}^n$, as follows. [1]

The *inverse* stereographic projection map from $\mathbb{R}^n$ to the unit sphere $S^n \subset \mathbb{R}^{n+1}$ sends a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ to the intersection point of the unit sphere $S^n \subset R^{n+1}$ with the line segment connecting $(x, 0) = (x_1, \ldots, x_n, 0)$ to the point $p = (0, \ldots, 0, 1)$. Formulaically, this is

$$\sigma \;:\; x \;\longrightarrow\; \Big(\frac{2x_1}{|x|^2+1}, \ldots, \frac{2x_n}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}\Big) \qquad\qquad (\text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n)$$

where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ as usual. The inverse map is

$$\sigma^{-1} \;:\; (y, z) \;=\; (y_1, \ldots, y_n, z) \;\longrightarrow\; \frac{y}{1-z} \;=\; \Big(\frac{y_1}{1-z}, \ldots, \frac{y_n}{1-z}\Big)$$

and this certifies that σ is a smooth homeomorphism of $\mathbb{R}^n$ with $S^n - \{p\}$. Certainly S^n is *compact*.

Thus, the corresponding extrinsic one-point compactification of $\mathbb{R}^n$ adjoins a point named ∞, and declares the neighborhoods of ∞ in $\mathbb{R}^n \cup \{\infty\}$ to be the inverse images $\sigma^{-1}(U - \{p\})$ of punctured neighborhoods $U - \{p\}$ of $p \in S^n$.

A *local basis* at ∞ consists of sets

$$\{\infty\} \cup \{x \in \mathbb{R}^n : |x| > r\} \qquad\qquad (\text{for } r \geq 0)$$

[1.1] Remark: A notable failing of this extrinsic stereographic compactification of $\mathbb{C} \approx \mathbb{R}^2$ is that it does not help describe the *complex structure* at the new point ∞, so that we have no immediate sense of functions' *holomorphy at infinity* or *meromorphy at infinity*.

[1] In general, a *one-point compactification* of a Hausdorff topological space X can be described *intrinsically*, without imbedding in a larger space and without comparison to a pre-existing compact space: let $\widetilde{X} = X \cup \{\infty\}$, and neighborhoods of ∞ are all sets in $\widetilde{X}$ of the form $\widetilde{X} - K$ where K is a compact subset of X, noting that Hausdorffness implies that compact sets are closed.

2. *The complex projective line* $\mathbb{CP}^1$

For purposes of complex analysis, a better description of a one-point compactification of $\mathbb{C}$ is an instance of the *complex projective space* $\mathbb{CP}^n$, a compact space containing $\mathbb{C}^n$, described as follows.

Let $\sim$ be the equivalence relation on $\mathbb{C}^{n+1} - \{0\}$ by $x \sim y$ when $x = \alpha \cdot y$ for some $\alpha \in \mathbb{C}^\times$. Thus, $x \sim y$ means that x and y lie on the same *complex line* inside $\mathbb{C}^{n+1}$. The complex projective n-space $\mathbb{CP}^n$ is the quotient of $\mathbb{C}^{n+1} - \{0\}$ by this equivalence relation:

$$\mathbb{CP}^n \quad = \quad \Big(\mathbb{C}^{n+1} - \{0\}\Big)\Big/\sim \quad \approx \quad \{\text{complex lines in } \mathbb{C}^{n+1}\}$$

Every equivalence class in $\mathbb{CP}^n$ has a representative in the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, and the further map to $\mathbb{CP}^n$ is *continuous*, so $\mathbb{CP}^n$ is *compact.*

There is the inclusion $\mathbb{C}^n \to \mathbb{CP}^n$ by

$$z = (z_1, \ldots, z_n) \quad \longrightarrow \quad \text{equivalence class of } (z_1, \ldots, z_n, 1) \quad = \quad \mathbb{C}^\times \cdot (z_1, \ldots, z_n, 1)$$

The image of $\mathbb{C}^n$ in $\mathbb{CP}^n$ misses exactly

$$\{(z_1, \ldots, z_n, 0)\}\Big/\sim \;\approx\; \mathbb{CP}^{n-1}$$

For $n = 1$, this is the single point

$$\infty \quad = \quad \{(z_1, 0)\}\Big/\sim \;\approx\; \mathbb{CP}^0 \quad \approx \quad \{\text{pt}\}$$

so $\mathbb{CP}^1$ is a one-point compactification of $\mathbb{C}$. Otherwise, $\mathbb{CP}^n$ is strictly bigger than a one-point compactification.

Homogeneous coordinates on $\mathbb{CP}^n$ are the coordinates on $\mathbb{C}^{n+1}$ for representatives of the quotient. Thus, for $\mathbb{C} \subset \mathbb{CP}^1$, the homogeneous coordinates for the image of z are $\begin{pmatrix} z \\ 1 \end{pmatrix}$, for example. Going in the other direction, given homogeneous coordinates $\begin{pmatrix} u \\ v \end{pmatrix}$, for $v \neq 0$, this represents the same equivalence class as does $\begin{pmatrix} u/v \\ 1 \end{pmatrix}$, which is the image of the point $u/v \in \mathbb{C}$. If $v = 0$, then necessarily $u \neq 0$, and $\begin{pmatrix} u \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is ∞, *the point at infinity.*

[2.1] Remark: This procedure gives $\mathbb{CP}^n$ a natural *complex structure* for all n, as illustrated in the $n = 1$ case just below, in contrast to the stereographic one-point compactification. However, even for $n = 1$, the meaning of *complex structure* will be considered at length only somewhat later, in discussion of (complex-) one-dimensional *complex manifolds*, also known as *Riemann surfaces.*

3. *Functions' behavior at infinity*

At least as a preliminary version, for a function f holomorphic in a region $|z| > r$

$$\begin{cases} \text{is holomorphic at } \infty & \Longleftrightarrow \quad z \to f(1/z) \text{ is holomorphic at } 0 \\ \text{is meromorphic at } \infty & \Longleftrightarrow \quad z \to f(1/z) \text{ is meromorphic at } 0 \\ \text{has an essential singularity at } \infty & \Longleftrightarrow \quad z \to f(1/z) \text{ has an essential singularity at } 0 \end{cases}$$

This is consistent with the one-point compactification's topology, declaring the neighborhoods of ∞ to be complements of compact subsets of $\mathbb{C}$ (with ∞ added), so mapping $z \to 1/z$ maps punctured neighborhoods of 0 to punctured neighborhoods of ∞, and vice-versa.

For example,

$$\begin{cases} \text{behavior of} \quad z \to z^2 \quad \text{at } \infty \iff \text{behavior of } z \to \frac{1}{z^2} \text{ at } 0 & \text{(meromorphic)} \\ \text{behavior of} \quad z \to \frac{1}{z^2} \quad \text{at } \infty \iff \text{behavior of } z \to z^2 \text{ at } 0 & \text{(holomorphic)} \\ \text{behavior of} \quad z \to \frac{z-1}{z+1} \quad \text{at } \infty \iff \text{behavior of } z \to \frac{\frac{1}{z}-1}{\frac{1}{z}+1} = \frac{1-z}{1+z} \text{ at } 0 & \text{(holomorphic)} \\ \text{behavior of} \quad z \to e^z \quad \text{at } \infty \iff \text{behavior of } z \to e^{1/z} = \ldots + \frac{1}{2z^2} + \frac{1}{z} + 1 \text{ at } 0 & \text{(ess sing)} \end{cases}$$

[3.1] Claim: The functions *holomorphic* on the whole $\mathbb{CP}^1$ are just constants. The functions f *meromorphic* on the whole $\mathbb{CP}^1$ are exactly *rational* functions $f(z) = \frac{P(z)}{Q(z)}$, with polynomials P, Q and Q not identically 0.

Proof: For f to be *holomorphic* at ∞ means that $z \to f(1/z)$ is holomorphic near 0. In particular, it is *bounded* on some neighborhood $|z| < \varepsilon$ of 0. Then $z \to f(z)$ is *bounded* on $|z| > 1/\varepsilon$. Certainly $z \to f(z)$ is bounded on the compact set $|z| \le \varepsilon$, so f is bounded and entire, so constant, by Liouville's theorem.

For f meromorphic at ∞, $z \to f(1/z)$ has a *finite-nosed* Laurent expansion at 0, convergent in some punctured neighborhood,

$$f(1/z) \;=\; \frac{c_N}{z^N} + \ldots + c_0 + c_1 z + \ldots \qquad \text{(for } 0 < |z| < \varepsilon\text{)}$$

On the compact set $|z| \le 1/\varepsilon$, f itself can have only finitely-many poles, say at $z_1, \ldots, z_n$, of orders $\nu_1, \ldots, \nu_n$. The function $g(z) = (z - z_1)^{\nu_1} \ldots (z - z_n)^{\nu_n} f(z)$ has *no* poles in $|z| \le 1/\varepsilon$, and $g(z)$ is meromorphic at ∞, since each $(z - z_j)^{\nu_j}$ is meromorphic at ∞. Then

$$g(z) \;=\; c_N z^N + \ldots + c_0 + \frac{c_1}{z} + \ldots \qquad \text{(for } |z| > 1/\varepsilon\text{)}$$

and $z^{-N} g(z)$ is *bounded* on $|z| > 1/\varepsilon$. The continuous function $|g(z)|$ is certainly bounded on the compact $|z| \le 1/\varepsilon$, so $|g(z)| \le B \cdot |z|^N$ for some B and N. As in the proof of Liouville's theorem, an entire function admitting such a bound is a polynomial of degree at most N. Thus, the original f was a rational function. ///

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4. *Linear fractional (Möbius) transformations*

The *general linear group* $GL_2(\mathbb{C})$ is the group of multiplicatively invertible two-by-two complex matrices. This group acts on two-by-one complex matrices $\mathbb{C}^2$ by matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \;=\; \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix}$$

The *linearity* of this action is that $g(c \cdot v) = c \cdot gv$ for $g \in GL_2(\mathbb{C})$, $c \in \mathbb{C}$, and $v \in \mathbb{C}^2$. In particular, the action of $GL_2(\mathbb{C})$ stabilizes the equivalence classes $\mathbb{C}^\times \cdot v$ used to form $\mathbb{CP}^1$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} ap + bq \\ cp + dq \end{pmatrix} \cdot \mathbb{C}^\times$$

On the image $\begin{pmatrix} z \\ 1 \end{pmatrix}$ of a point $z \in \mathbb{C}$ in $\mathbb{CP}^1$, in *homogeneous coordinates*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \;=\; \begin{pmatrix} az+b \\ cz+d \end{pmatrix}$$

In the typical case that $cz+d \neq 0$,

$$\begin{pmatrix} az+b \\ cz+d \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \cdot (cz+d) \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix} \cdot \mathbb{C}^\times$$

That is, the point $z \in \mathbb{C} \subset \mathbb{CP}^1$ is mapped to $\frac{az+b}{cz+d} \in \mathbb{C} \subset \mathbb{CP}^1$ when $cz+d \neq 0$. When $cz+d = 0$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} az+b \\ cz+d \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} az+b \\ 0 \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \infty$$

Write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) \;=\; \frac{az+b}{cz+d}$$

with the implicit qualification that the image is ∞ when $cz+d = 0$.

We can see where the point ∞ is mapped:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\infty) \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} a \\ c \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{cases} \begin{pmatrix} \frac{a}{c} \\ 1 \end{pmatrix} \cdot \mathbb{C}^\times & (\text{when } c \neq 0) \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \mathbb{C}^\times & (\text{when } c = 0) \end{cases}$$

That is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\infty) \;=\; \begin{cases} \frac{a}{c} & (\text{when } c \neq 0) \\ \infty & (\text{when } c = 0) \end{cases}$$

The *continuity* of the action of $GL_2(\mathbb{C})$ on $\mathbb{C}^2$ results in the continuity of the action of $GL_2(\mathbb{C})$ on $\mathbb{CP}^1$.

[4.1] Remark: Similarly, $GL_n(\mathbb{C})$ acts by generalized linear fractional transformations on $\mathbb{CP}^{n-1}$, by

$$\begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} \cdot \mathbb{C}^\times \;=\; \begin{pmatrix} g_{11}\omega_1 + \ldots + g_{1n}\omega_n \\ \vdots \\ g_{n1}\omega_1 + \ldots + g_{nn}\omega_n \end{pmatrix} \cdot \mathbb{C}^\times$$

[4.2] Claim: The holomorphic automorphisms of $\mathbb{CP}^1$, that is, the meromorphic functions f on $\mathbb{C}$ also meromorophic at infinity, and have inverse maps of the same sort, are exactly the linear fractional transformations.

Proof: From above, $f(z) = P(z)/Q(z)$ for polynomials P, Q, with Q not identically 0. Without loss of generality, we can suppose P, Q are relatively prime in the (Euclidean) ring $\mathbb{C}[X]$. If both are constant, then f is constant, contradicting injectivity.

If Q has positive degree, then it has a zero z_o, and $f(z_o) = \infty$. Let γ be a linear fractional transformation mapping $\infty \to z_o$. Replacing f by $f \circ \gamma$, the modified f maps $\infty \to \infty$. No other point can be mapped to ∞, by injectivity, so this modified f is be a polynomial.

If the degree of f is greater than 1 and if f has two or more distinct complex zeros, it maps those two points to 0, contradicting injectivity. Thus, $f(z) = c(z - z_o)^n$ for some non-zero c and for some $1 \leq n \in \mathbb{Z}$. But this maps $z_o + \mu$ to 1 for all n^{th} roots of unity μ, contradicting injectivity if $n > 1$. Thus, the modified f is linear, and is a linear fractional transformation. Thus, the original f was a linear fractional transformation. ///
