

## 08. Conformal mapping

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is  
http://www.math.umn.edu/~garrett/m/complex/notes\_2020-21/08\_conformal\_mapping.pdf]

1. Conformal (angle-preserving) maps
2. Lines and circles and linear fractional transformations
3. Elementary examples
4.  $f'(z_o) = 0$  implies local non-injectivity
5. Automorphisms of the disk and of  $\mathfrak{H}$
6. Schwarz' lemma
7. Riemann Mapping Theorem

### 1. Conformal (angle-preserving) maps

A complex-valued function  $f$  on a non-empty open  $U \subset \mathbb{C}$  is *conformal* if it *preserves angles*, in the sense that, for any two smooth parametrized curves  $\gamma : [a, b] \rightarrow \mathbb{C}$ ,  $\delta : [c, d] \rightarrow \mathbb{C}$  with  $\gamma(a) = z_o = \delta(c)$ , the angle between  $\gamma'(a)$  and  $\delta'(c)$  is equal to the angle between  $(f \circ \gamma)'(a)$  and  $(f \circ \delta)'(c)$ . The function  $f$  is *orientation-preserving* if the *directed* angle is preserved.

Explicitly, for non-zero  $z, w \in \mathbb{C}$ ,

$$(\text{directed angle from } z \text{ to } w) = \theta \in \mathbb{R}, \text{ such that } \frac{w}{|w|} = e^{i\theta} \frac{z}{|z|}$$

That is,

$$e^{i\theta} = \frac{w}{|w|} \Big/ \frac{z}{|z|}$$

[1.1] **Claim:** A holomorphic function  $f$  is *conformal* and *orientation-preserving* at points  $z_o$  where  $f'(z_o) \neq 0$ .

*Proof:* This is a direct computation, using the chain rule, noting that  $f'$  is the *complex* derivative of  $f$ , while  $\gamma'$  is the *real* derivative. With  $\gamma(a) = z_o = \delta(c)$ ,

$$(f \circ \gamma)'(a) = f'(\gamma(a)) \cdot \gamma'(a) = f'(z_o) \cdot \gamma'(a) \qquad (f \circ \delta)'(c) = f'(\delta(a)) \cdot \delta'(a) = f'(z_o) \cdot \delta'(a)$$

so

$$\frac{(f \circ \gamma)'(a)}{|(f \circ \gamma)'(a)|} \Big/ \frac{(f \circ \delta)'(a)}{|(f \circ \delta)'(a)|} = \frac{f'(z_o) \gamma'(a)}{|f'(z_o) \gamma'(a)|} \Big/ \frac{f'(z_o) \delta'(a)}{|f'(z_o) \delta'(a)|} = \frac{\gamma'(a)}{|\gamma'(a)|} \Big/ \frac{\delta'(a)}{|\delta'(a)|}$$

showing that directed angles are preserved. ///

[1.2] **Remark:** Holomorphic  $f$  on  $U$  with non-vanishing derivative maps the mutually orthogonal families of lines  $\text{Re}(z) = x = \text{const}$  and  $\text{Im}(z) = y = \text{const}$  to mutually orthogonal curves.

For example, letting  $f(z) = z^2$  on the upper half-plane, the lines  $x \rightarrow x + iy_o$  become parabolas  $x \rightarrow x^2 - 2ixy_o - y_o^2$  opening horizontally, and the lines  $y \rightarrow x_o + iy$  become parabolas  $y \rightarrow -y^2 + 2x_o y + x_o^2$  opening vertically.

Letting  $f(z) = \sqrt{z}$  on  $\mathbb{C} - [0, +\infty)$  amounts to looking at inverse images in the previous example, giving mutually orthogonal lines  $\text{Re}(z^2) = \text{const}$  and  $\text{Im}(z^2) = \text{const}$ , namely, hyperbolas  $x^2 - y^2 = \text{const}$  and hyperbolas  $xy = \text{const}$ .

## 2. Lines and circles and linear fractional transformations

[2.1] **Theorem:** The collection of lines and circles in  $\mathbb{C} \cup \{\infty\}$  is stabilized by linear fractional transformations, and is acted upon *transitively* by them.

*Proof:* Clearly *affine* maps  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z) = (az + b)/d$ , with  $ad \neq 0$ , which are combinations of translations, rotations, and dilations, preserve lines and circles. Given this, a group-theoretic result greatly simplifies things:

[2.2] **Claim:** (*Bruhat decomposition*) Let  $P$  be the upper-triangular matrices in  $GL_2(\mathbb{C})$ , and the *Weyl element*  $w_o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . Then

$$GL_2(\mathbb{C}) = P \sqcup Pw_oP \quad (\text{disjoint union})$$

*Proof:* A group element  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is in  $P$  if  $c = 0$ , so consider  $c \neq 0$ . Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in P$$

That is, for  $g \notin P$ ,  $gPw_o \cap P \neq \emptyset$ . That is,  $gP \cap Pw_o \neq \emptyset$ , so  $gP \cap Pw_oP \neq \emptyset$ , and  $g \in Pw_oP$ . ///

Invoking the Bruhat decomposition, since  $g \in P$  preserves lines and circles, it suffices to prove that the Weyl element  $w_o$  preserves lines and circles. The real and imaginary parts of  $w_o(z) = 1/z$  are easily observed:

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

so

$$\operatorname{Re}(1/z) = \frac{x}{|z|^2} \quad \operatorname{Im}(1/z) = -\frac{y}{|z|^2}$$

It is convenient that  $w_o^2$  gives the identity map, so the images and inverse images of sets under  $w_o$  are the same things. Given a line  $ax + by = c$  with real  $a, b, c$ , the image of the line is

$$a \frac{x}{|z|^2} - b \frac{y}{|z|^2} = c$$

or

$$ax - by = cx^2 + cy^2$$

giving the circle

$$\left(x - \frac{a}{c}\right)^2 + \left(y + \frac{b}{c}\right)^2 = \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2$$

The image of a circle  $|z - z_o| = r$  is

$$\left|\frac{1}{z} - z_o\right| = r$$

or  $|1 - z_o \cdot z| = r \cdot |z|$  or

$$1 - 2 \operatorname{Re}(z_o \cdot z) + |z_o|^2 \cdot |z|^2 = r^2 \cdot |z|^2$$

In the special case that  $|z_o| = r$ , this is a line. Otherwise, it is a circle.

Three points on a circle determine the circle completely. A line in  $\mathbb{C}$  can be viewed determined by two points on it in  $\mathbb{C}$ , and inevitably passing through  $\infty$ . The group of linear fractional transformation actions is *triply transitive* in the sense that it can map any triple of distinct points to any other, so is transitive on circles-and-lines. ///

[2.3] Remark: Beware: except for *affine* maps  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z) = (az + b)/d$ , with  $ad \neq 0$ , linear fractional transformations do *not* respect *centers* of circles.

---

### 3. Elementary examples

A *sector* is an open set of the form

$$\{z = re^{i\theta} : r > 0, a < \theta < b, \text{ with } |b - a| < 2\pi\}$$

Two sectors of the special form

$$\{z = re^{i\theta} : r > 0, 0 < \theta < b, \text{ with } b < 2\pi\} \quad \{z = re^{i\theta} : r > 0, 0 < \theta < b', \text{ with } b' < 2\pi\}$$

can be mapped holomorphically to each other by a *power map*

$$z \longrightarrow z^{\frac{b'}{b}} = e^{\frac{b'}{b} \cdot \log z}$$

with a continuous logarithm defined on the first sector by

$$\log z = i\pi + \int_1^{z/e^{i\pi}} \frac{dw}{w}$$

integrating along a straight line segment. In polar coordinates, this is simply

$$re^{i\theta} \longrightarrow r^{\frac{b'}{b}} e^{i\theta \frac{b'}{b}}$$

Exhibiting the map as a *holomorphic* map shows that it preserves angles.

Sectors with edges elsewhere than the positive real axis can be rotated, by map  $z \rightarrow \mu \cdot z$  with  $|\mu| = 1$ , to put either edge on the positive real axis. Thus, the problem of mapping one sector to another reduces to that simpler case, by pre-composing and post-composing with rotations:

$$\{z = re^{i\theta} : r > 0, a < \theta < b, \text{ with } |b - a| < 2\pi\}$$

can be mapped to another sector

$$\{z = re^{i\theta} : r > 0, a' < \theta < b', \text{ with } |b' - a'| < 2\pi\}$$

by

$$z \longrightarrow \frac{z}{e^{ia}} \longrightarrow \left(\frac{z}{e^{ia}}\right)^{\frac{b'-a'}{b-a}} \longrightarrow e^{ia'} \cdot \left(\frac{z}{e^{ia}}\right)^{\frac{b'-a'}{b-a}}$$

Again, there is an unambiguous choice of continuous  $\alpha^{\text{th}}$  power on such a sector, by

$$z^\alpha = e^{\alpha \cdot \log z}$$

where  $\log z$  is defined on  $\mathbb{C}$  with any ray  $\{re^{i\theta_0} : r > 0\}$  removed, with this ray not lying in the given sector. Again, such a logarithm can be defined by

$$\log z = i(\theta_0 + \pi) + \int_1^{z/e^{i(\theta_0+\pi)}} \frac{dw}{w}$$

integrating along a straight line segment from 1 to  $z/e^{i(\theta_0+\pi)}$ . That is, all sectors are *conformally equivalent*.

In other words, while  $z \rightarrow z^\alpha$  with real  $\alpha \neq 0$  is *conformal* at every point other than  $z = 0$ , it alters angles at 0 by multiplying angles by  $\alpha$ .

A *bigon* is an open subset of  $\mathbb{CP}^1$  bounded by two *arcs*, by which we for present purposes we mean either straight line segments, possibly infinite in one or both directions, and/or arcs of circles with angle measure  $< 2\pi$ .

The non-degenerate cases are specified by two vertices  $z_1, z_2 \in \mathbb{PC}^1$ , and two distinct circles-or-lines passing through both  $z_1, z_2$ . This configuration cuts  $\mathbb{PC}$  into four connected components, each of which is a bigon.

In the degenerate case that the two points are identical,  $\mathbb{CP}^1$  is cut into only three connected components.

[3.1] **Claim:** All bigons in the non-degenerate case are conformally equivalent, and are conformally equivalent to the upper half-plane.

*Proof:* Given vertices  $z_1 \neq z_2$ , map  $z_1 \rightarrow 0$  and  $z_2 \rightarrow \infty$  by a linear fractional transformation. Since linear fractional transformations preserve lines and circles, the boundary of the image consists of two straight lines from 0 to  $\infty$ , so the image is a *sector*. *Rotate* the sector until it is of the form  $\{re^{i\theta} : r > 0, 0 < \theta < b, \text{ with } b < 2\pi\}$ , and then apply a *power map* to obtain the upper half-plane  $\mathfrak{H}$ . Thus, any non-degenerate bigon is conformally equivalent to the upper half-plane, so they are equivalent to each other. ///

[3.2] **Claim:** All degenerate bigons are conformally equivalent, and are conformally equivalent to the strip  $0 < \text{Re}(z) < 1$ .

*Proof:* Map the single vertex to  $\infty$ . The two bounded arcs, mapped to arcs passing through  $\infty$ , must become straight lines with no other intersections, thus, parallel. Translate, rotate and dilate to obtain the lines  $\text{Re}(z) = 0$  and  $\text{Re}(z) = 1$ . ///

[3.3] **Remark:** An open disk can be considered as a non-degenerate bigon in many ways, as can a half-plane. *The Cayley map*

$$z \longrightarrow \frac{z+i}{iz+1}$$

maps  $i \rightarrow \infty$ ,  $-i \rightarrow 0$ , and  $1 \rightarrow 1$ , so maps the unit circle to the real line, because linear fractional transformations preserve circles-and-lines. Thus, it maps the connected components of the complement of the circle to the connected components of the complement of the real line: since  $0 \rightarrow i$ , the *interior* of the circle is mapped to the *upper* half-plane.

## 4. $f'(z) = 0$ implies local non-injectivity

The following generally-useful corollary of the argument principle could have been proven earlier, but is perhaps of interest here as a sort of converse to the holomorphic inverse-function theorem:

[4.1] **Claim:** A non-constant holomorphic function  $f$  near  $z_o$  vanishing to order exactly  $k$  at  $z_o$  is  $(k+1)$ -to-1 in every sufficiently small punctured disk at  $z_o$ , in the following precise sense. Given a sufficiently small  $r > 0$ , there is a neighborhood  $U$  of  $z_o$  such that, for all  $z_1$  in  $U$ , the value  $f(z_1)$  is hit  $k+1$  times inside  $|z - z_o| = r$ .

*Proof:* The *idea* is that, on one hand, since  $f(z) - f(z_o)$  vanishes to order  $k$  at  $z_o$ , the argument principle would give

$$k + 1 \leq \frac{1}{2\pi i} \int_{\gamma} \frac{(f(w) - f(z_o))' dw}{f(w) - f(z_o)} = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(w) dw}{f(w) - f(z_o)}$$

where  $\gamma$  is the counterclockwise path around  $|z - z_o| = r$ . On the other hand,

$$z \rightarrow \frac{1}{2\pi i} \int_{\gamma} \frac{f'(w) dw}{f(w) - f(z)}$$

should be a holomorphic function of  $z$  on some neighborhood  $U$  of  $z_o$ . An integer-valued holomorphic function is constant. Thus, it should be every value  $f(z_1)$  for  $z_1 \in U$  is hit at least  $k+1$  times inside the circle  $|z - z_o| = r$ .

Two details are missing. First, we want to be sure that the denominator  $f(w) - f(z_o)$  does not vanish on the circle  $|z - z_o| = r$ . We claim that we can shrink  $r$  if necessary so that  $f(w) \neq f(z_o)$  on that circle: indeed, if there were a sequence of points  $z_1, z_2, \dots$  approaching  $z_o$  with  $f(z_j) = f(z_o)$ , then by the identity principle  $f$  is constant. Second, we want the denominator  $f(w) - f(z)$  to not vanish for all  $w$  on the circle and for all  $z$  sufficiently near to  $z_o$ . Indeed, the set of images  $I = \{f(z) : |z - z_o| = r\}$  is a continuous image of a compact set, so is compact, so is *closed*. Since  $f(z_o) \notin I$ , there is some neighborhood  $N$  of  $f(z_o)$  disjoint from  $I$ . The inverse image  $f^{-1}(N)$  is open, by continuity, and contains  $z_o$ , so there is a neighborhood  $U$  of  $z_o$  such that  $f(U) \cap I = \emptyset$ . Shrink  $U$  to be an open disk at  $z_o$ , so it is connected. ///

## 5. Automorphisms of the disk and of $\mathfrak{H}$

First, we demonstrate some explicit groups of linear transformations stabilizing the upper half-plane, or stabilizing the unit disk, in both cases large enough to act *transitively*. Then we invoke Schwarz' lemma (from the following section) to see that these groups are *all* the holomorphic automorphisms of these regions.

[5.1] **Claim:** The linear fractional transformations arising from

$$SL_2(\mathbb{R}) = \{\text{two-by-two real matrices with determinant} = 1\}$$

stabilize the upper half-plane  $\mathfrak{H}$ , and act transitively on it. In particular,  $\text{Im} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) (z) = \frac{\text{Im}(z)}{|cz + d|^2}$ .

*Proof:* First,  $SL_2(\mathbb{R})$  is a subgroup of  $GL_2(\mathbb{R})$ , including inverses. Directly compute the effect on imaginary parts:

$$\begin{aligned} 2i \text{Im} \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) (x + iy) &= 2i \text{Im} \frac{az + b}{cz + d} = \frac{az + b}{cz + d} - \frac{a\bar{z} + b}{c\bar{z} + d} = \frac{(az + b)(c\bar{z} + d) - (a\bar{z} + b)(cz + d)}{|cz + d|^2} \\ &= \frac{(acz\bar{z} + bc\bar{z} + adz + bd) - (acz\bar{z} + bcz + ad\bar{z} + bd)}{|cz + d|^2} = \frac{(ad - bc)(z - \bar{z})}{|cz + d|^2} = \frac{z - \bar{z}}{|cz + d|^2} = \frac{2i \text{Im}(z)}{|cz + d|^2} \end{aligned}$$

This shows that  $SL_2(\mathbb{R})$  stabilizes the upper half-plane. To show transitivity, observe that for  $x + iy \in \mathfrak{H}$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} (i) = x + iy$$

so the point  $i \in \mathfrak{H}$  can be mapped to any other. ///

[5.2] Remark: The *special orthogonal group*

$$SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} : \theta \in \mathbb{R} \right\} \subset SL_2(\mathbb{R})$$

is the stabilizer subgroup in  $SL_2(\mathbb{R})$  of the point  $i \in \mathfrak{H}$ .

Let  $g \rightarrow g^*$  be conjugate transpose:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$$

and put  $S = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

[5.3] Claim: The subgroup

$$SU(1,1) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) : g^* S g = S, \det g = 1 \right\} \subset GL_2(\mathbb{C})$$

stabilizes the open unit disk and acts transitively on it.

*Proof:* Observe that

$$-|\alpha|^2 + |\beta|^2 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^* S \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

For  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$ ,

$$\begin{aligned} (|gz|^2 - 1) \cdot |cz + d|^2 &= \left( \left| \frac{az + b}{cz + d} \right|^2 - 1 \right) \cdot |cz + d|^2 = |az + b|^2 - |cz + d|^2 \\ &= \begin{pmatrix} az + b \\ cz + d \end{pmatrix}^* S \begin{pmatrix} az + b \\ cz + d \end{pmatrix} = \left( g \begin{pmatrix} z \\ 1 \end{pmatrix} \right)^* S \left( g \begin{pmatrix} z \\ 1 \end{pmatrix} \right) = \begin{pmatrix} z \\ 1 \end{pmatrix}^* g^* S g \begin{pmatrix} z \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} z \\ 1 \end{pmatrix}^* S \begin{pmatrix} z \\ 1 \end{pmatrix} = |z|^2 - 1 < 0 \quad (\text{for } z \text{ in the unit disk}) \end{aligned}$$

This proves that  $U(1,1)$  stabilizes the open disk. There are rotations  $z \rightarrow \mu^2 \cdot z$  given by  $\begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$  in  $SU(1,1)$  for  $|\mu| = 1$ , and these are transitive on each circle of radius  $0 \leq r < 1$ . The elements

$$\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

are in  $SU(1,1)$ , and send

$$0 \longrightarrow \tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{e^{2t} - 1}{e^{2t} + 1} \quad (\text{for } t \in \mathbb{R})$$

By the intermediate value theorem, since  $\lim_{t \rightarrow +\infty} \tanh t = 1$ , every real value  $T$  in the interval  $0 \leq T < 1$ . Thus,  $SU(1, 1)$  is transitive on the open unit disk. ///

[5.4] **Remark:** Conjugation by the Cayley map  $C = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$  has the property that  $C \cdot g \cdot C^{-1} \in SL_2(\mathbb{R})$  for  $g \in SU(1, 1)$ , and *vice-versa*. In particular,  $C^{-1}$  conjugates rotations of the disk to the group  $SO_2(\mathbb{R})$ .

[5.5] **Corollary:** (of Schwarz' lemma) Any holomorphic automorphism of the open unit disk is given by an element of  $SU(1, 1)$ . Any holomorphic automorphism of the upper half-plane is given by an element of  $SL_2(\mathbb{R})$ .

*Proof:* Given a holomorphic map  $f$  of the open disk to itself, compose with an element of  $U(1, 1)$  to adjust so that  $f(0) = 0$ . Certainly  $|f(z)| \leq 1$  for  $|z| < 1$ . By Schwarz' lemma,  $|f(z)| \leq |z|$  and  $|f'(0)| \leq 1$ , and if equality holds at any point with  $0 < |z| < 1$  or if  $|f'(0)| = 1$ , then  $f(z) = \mu \cdot z$  with  $|\mu| = 1$ .

By the holomorphic inverse function theorem,  $(f^{-1})'(0) = 1/f'(0)$ . Also, Schwarz' lemma applies to  $f^{-1}$ . Thus,

$$1 \leq \frac{1}{|f'(0)|} = |(f^{-1})'(0)| \leq 1$$

implies equality, and that  $f(z) = \mu \cdot z$  for some  $|\mu| = 1$ . This shows that the linear fractional transformations given by  $SU(1, 1)$  are the whole holomorphic automorphism of the open unit disk.

Similarly, for a holomorphic automorphism  $f$  of  $\mathfrak{H}$ , let  $f(i) = z$ , and let  $g \in SL_2(\mathbb{C})$  map  $g(z) = i$ . Then

$$h = C^{-1} \circ g \circ f \circ C$$

is a holomorphic automorphism of the open unit disk fixing 0, so is a rotation coming from  $SU(1, 1)$ . Note that  $CgC^{-1} \in SO_2(\mathbb{R})$ . Then

$$f = g \circ (C \circ h \circ C^{-1})$$

expresses  $f : \mathfrak{H} \rightarrow \mathfrak{H}$  as a composition of linear fractional transformations  $g \in SL_2(\mathbb{R})$ . ///

[5.6] **Remark:** The action on  $\mathbb{CP}^1$  given by *scalar* elements of  $GL_2(\mathbb{C})$  is *trivial*, so the central subgroups of  $U(1, 1)$  and of  $SL_2(\mathbb{R})$  act trivially.

## 6. Schwarz' lemma

[6.1] **Theorem:** For  $f$  holomorphic on  $|z| < 1$ , with bound  $|f(z)| \leq 1$ , and with  $f(0) = 0$ ,

$$|f(z)| \leq |z| \quad \text{and} \quad |f'(0)| \leq 1$$

Equality  $|f(z_o)| = |z_o|$  holds for some  $0 < |z_o| < 1$  if and only if  $f(z) = \mu \cdot z$  for some constant  $\mu$  with  $|\mu| = 1$ .

*Proof:* The function  $F(z) = f(z)/z$  has a removable singularity at  $z = 0$ , and takes value  $f'(0)$  there. For each  $0 < r < 1$ ,

$$|F(z)| = \frac{|f(z)|}{r} \leq \frac{1}{r} \quad (\text{on the circle } |z| = r)$$

By the maximum modulus principle,  $|F(z)| \leq 1/r$  on  $|z| \leq r$ . Thus, for all  $r'$  with  $0 < r < r' < 1$ ,  $|F(z)| \leq 1/r'$  on  $|z| \leq r$ , so  $|F(z)| \leq 1$  on  $|z| \leq r$ . This holds for every  $0 < r < 1$ , so  $|F(z)| \leq 1$  on  $|z| < 1$ . This already gives  $|f'(0)| \leq 1$ .

If  $|F(z_o)| = 1$  for some  $0 < |z_o| < 1$ , or if  $|f'(0)| = 1$ , then  $|F|$  attains its sup in the interior, so is a constant  $\mu$ , and  $f(z) = \mu \cdot z$ . ///

## 7. Riemann Mapping Theorem

[7.1] **Theorem:** Given a non-empty, simply-connected open subset  $\Omega \subset \mathbb{C}$  that is not the whole complex plane  $\mathbb{C}$ , there is a holomorphic bijection of  $\Omega$  to the open unit disk.

[7.2] **Remark:** Riemann's original argument used a version of *Dirichlet's principle*, an assertion that functionals are minimized/maximized on reasonable subsets of spaces of functions, which is not exactly true. It was not until 1906 that Giuseppe ("Beppo") Levi proved a correct version, using Hilbert spaces rather than Banach spaces. G. Frobenius in 1907 showed similar ideas. These were the forerunners of (Hilbert-space) Sobolev spaces of 30 years later.

*Proof:* [... iou ...]

///

The special case of *polygonal* regions allows explicit formulaic mapping.

Finitely-multiply-connected regions can be conformally mapped to *disks with slits* (*schlicht regions*) at various radiuses and of various arc-width, with some uniqueness assertions.

---