(*January 31, 2021*)

09. *Product expansion of* $\sin x$

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[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2020-21/09_product_sine.pdf]

1. Partial fraction expansion $\dfrac{1}{\sin^2 x} = \sum_{n\in\mathbb{Z}} \dfrac{1}{(x-\pi n)^2}$
2. Partial fraction expansion $\cot x = \dfrac{1}{z} + \sum_{n\ge 1} \Big(\dfrac{1}{z-n} + \dfrac{1}{z+n}\Big)$
3. Product expansion $\sin x = x \cdot \prod_{n\ge 1} \Big(1 - \dfrac{x^2}{\pi^2 n^2}\Big)$

We might want a meromorphic function f on $\mathbb{C}$ with poles at a prescribed set $\{z_1, z_2, \ldots\} \subset \mathbb{C}$. More precisely, we might prescribe *pole data* at each pole z_j, namely, the negative-index Laurent coefficients at z_j: for each pole z_j, specify $0 < n \in \mathbb{Z}$ and coefficients $c_{-n}, c_{-n+1}, \ldots, c_{-1}$ and require

$$f(z) \;=\; \frac{c_{-n}}{(z-z_j)^n} + \frac{c_{-n+1}}{(z-z_j)^{n-1}} + \ldots + \frac{c_{-1}}{z-z_j} + (\text{holomorphic at } z_j)$$

Postponing the general case, we give an illustrative example of special interest. As late as 1735, no one knew a simpler expression for

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots$$

This was the *Basel problem*, after the Swiss home of the Bernoullis, a dominant force in European mathematics at the time. L. Euler solved the problem, winning notoriety at an early age, and introducing a new idea, as follows. Given non-zero numbers $a_1, \ldots, a_n$, a polynomial with constant term 1 having these numbers as roots is

$$\Big(1 - \frac{x}{a_1}\Big)\Big(1 - \frac{x}{a_2}\Big)\ldots\Big(1 - \frac{x}{a_n}\Big) \;=\; 1 - (\frac{1}{a_1} \ldots + \frac{1}{a_n})x + (\frac{1}{a_1 a_2} + \frac{1}{a_1 a_3} + \frac{1}{a_2 a_3} + \ldots)x^2 + \ldots + (-1)^n \frac{x^n}{a_1 \ldots a_n}$$

Imagine that $\frac{\sin \pi x}{\pi x}$ has an analogous product expansion, in terms of its zeros at $\pm 1, \pm 2, \pm 3, \ldots$, up to a normalizing constant needing determination: using the power series expansion of $\sin x$,

$$\frac{(\pi x) - \frac{(\pi x)^3}{3!} + \ldots}{\pi x} \;=\; \frac{\sin \pi x}{\pi x} \;=\; C \cdot \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2}) \;=\; C \cdot \Big(x - x^3 \sum_n \frac{1}{n^2} + \ldots\Big)$$

Assuming this works, equating constant terms gives $C = 1$:

$$\frac{\sin \pi x}{\pi x} \;=\; \prod_{n\ge 1} \Big(1 - \frac{x^2}{n^2}\Big)$$

Equating coefficients of x^2 gives

$$\frac{\pi^2}{6} \;=\; \sum \frac{1}{n^2}$$

Euler only proved this product expansion years later, but the *heuristic* won him considerable notoriety, because it suggested an *underlying causal mechanism.* Further, once observed, the plausibility of the heuristic is easy to verify *numerically.*

The product expansion does not overtly mention complex numbers, but its simplest verification is an application of basic *complex analysis*: *Liouville's theorem* and *Laurent expansions* near poles. [1]

[1] Weierstraß and Hadamard product expansions apply to general *entire* functions, but with more overhead than needed here.

1. *Partial fraction expansion* $\frac{1}{\sin^2 x} = \sum_{n\in\mathbb{Z}} \frac{1}{(x-\pi n)^2}$

We claim that there is a *partial fraction expansion*

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \qquad \text{or, equivalently,} \qquad \frac{1}{\sin^2 z} \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z-\pi n)^2}$$

First, note that the indicated infinite sums do converge absolutely, *uniformly on compacts* away from the poles, so give holomorphic functions away from their poles.

Both sides of the (first) alleged equality have double poles exactly at integers. The Laurent expansion of the right-hand side near $n \in \mathbb{Z}$ begins

$$\frac{1}{(z-n)^2} + \text{holomorphic}$$

The left-hand side is *periodic*, so it suffices to see the Laurent expansion near 0:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{\pi^2}{\left(\pi z + \frac{(\pi z)^3}{3!} + \ldots\right)^2} \;=\; \frac{1}{z^2} \cdot \frac{1}{\left(1 - \frac{\pi^2 z^2}{3!} + \ldots\right)^2} \;=\; \frac{1}{z^2} \cdot \left(1 + \frac{\pi^2 z^2}{3!} + \ldots\right)^2 \;=\; \frac{1}{z^2} + \text{holomorphic}$$

This Laurent expansion matches that of the partial fraction expansion. Thus,

$$f(z) \;=\; \frac{\pi^2}{\sin^2 \pi z} - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$$

has no poles in $\mathbb{C}$, so is *entire*. On the real line, after cancellation of poles, f *continuous*. The periodicity $f(z+1) = f(z)$ is visible, so f is *bounded* on the real line. In fact, since f is bounded on any region $\{x+iy : 0 \le x \le 1,\ |y| \le N\}$, the periodicity gives the boundedness of $f(z)$ on every band $|y| \le N$ containing $\mathbb{R}$.

Both parts of $f(z)$ go to 0 as $|y| \to \infty$. Thus, f is bounded and entire, so constant, by *Liouville's theorem.* Since $f(z) \to 0$ as $|y| \to \infty$, this constant is 0. ///

2. *Partial fraction expansion* $\cot x = \frac{1}{z} + \sum_{n\ge 1} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$

Regroup the partial fraction expansion of $\pi^2/\sin^2 \pi x$ to improve convergence:

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \frac{1}{z^2} + \sum_{n\ge 1} \left(\frac{1}{(z-n)^2} + \frac{1}{(z+n)^2}\right)$$

The left-hand side is the derivative of $-\pi \cot \pi z$, and with the improved convergence the right-hand side is the obvious termwise derivative: up to a constant C,

$$\pi \cot \pi z \;=\; C + \frac{1}{z} + \sum_{n\ge 1} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)$$

The identity

$$\frac{1}{z-n} + \frac{1}{z+n} \;=\; \frac{(z+n)+(z-n)}{z^2-n^2} \;=\; \frac{2z}{z^2-n^2}$$

certifies that convergence is uniform and absolute, *and* that the summands are *odd* functions of z. Everything but the constant C is *odd* as a function of z, so $C = 0$. Thus,

$$\pi \cot \pi z \;=\; \frac{1}{z} + \sum_{n \geq 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

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3. *Product expansion* $\sin x = x \cdot \prod_{n \geq 1} (1 - \frac{x^2}{\pi^2 n^2})$

Also, $\pi \cot \pi z$ is the logarithmic derivative of $\sin \pi z$:

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{\pi \cos \pi z}{\sin \pi z} \;=\; \pi \cot \pi z$$

Thus,

$$\frac{d}{dz} \log(\sin \pi z) \;=\; \frac{(\sin \pi z)'}{\sin \pi z} \;=\; \frac{1}{z} + \sum_{n \geq 1} \Big(\frac{1}{z-n} + \frac{1}{z+n} \Big)$$

We intend to integrate. First, anticipating our goal, note that

$$\frac{d}{dz} \log \big(1 - \frac{z}{n}\big) \;=\; \frac{-\frac{1}{n}}{1 - \frac{z}{n}} \;=\; \frac{1}{z-n}$$

Thus, integrating, for some constant C,

$$\log(\sin \pi z) \;=\; C + \log z + \sum_{n \geq 1} \Big(\log \big(1 - \frac{z}{n}\big) + \log \big(1 + \frac{z}{n}\big) \Big) \;=\; C + \log z + \sum_{n \geq 1} \log \Big(1 - \frac{z^2}{n^2}\Big)$$

Exponentiating,

$$\sin \pi z \;=\; e^C \cdot z \cdot \prod_{n \geq 1} \Big(1 - \frac{z^2}{n^2}\Big)$$

Looking at the power series at $z = 0$, we see that $e^C = \pi$, so

$$\sin \pi z \;=\; \pi z \cdot \prod_{n \geq 1} \Big(1 - \frac{z^2}{n^2}\Big)$$
