(*February 3, 2021*)

11. Harmonic functions, Poisson kernels

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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The two-dimensional Euclidean *Laplacian* is

$$\Delta \;=\; \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Many of these ideas are not specific to two dimensions: the Laplacian on $\mathbb{R}^n$ is [1]

$$\Delta \;=\; \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

A twice-continuously-differentiable complex-valued function u on a non-empty open set $U \subset \mathbb{C} \approx \mathbb{R}^2$ satisfying *Laplace's equation* $\Delta u = 0$ is also called *harmonic*. In some contexts, a harmonic function is understood to be *real*-valued.

With the notation

$$\frac{\partial}{\partial z} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad\qquad \frac{\partial}{\partial \overline{z}} \;=\; \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

we have

$$\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \overline{z}} \;=\; \frac{\partial}{\partial \overline{z}} \circ \frac{\partial}{\partial z} \;=\; \frac{1}{4}\cdot\Delta$$

A holomorphic function u satisfies the Cauchy-Riemann equation $\partial u/\partial\overline{z} = 0$, so *every holomorphic function is harmonic.* Similarly, every *every conjugate-holomorphic function is harmonic.* Thus, for holomorphic f, the real and imaginary parts

$$\mathrm{Re}\big(f(z)\big) \;=\; \tfrac{1}{2}\Big(f(z) + \overline{f(z)}\Big) \qquad\qquad \mathrm{Im}\big(f(z)\big) \;=\; \frac{1}{2i}\Big(f(z) - \overline{f(z)}\Big)$$

are *harmonic*, and real-valued.

Harmonic functions have a mean-value property similar to holomorphic functions. This yields the *Poisson formula*, recovering interior values from boundary values, much as Cauchy's formula does for holomorphic functions. The solution of the *Dirichlet problem* is a converse: *every* function on the boundary of a disk arises as the boundary values of a harmonic function on the disk. We prove this for relatively nice functions on the boundary, but with adequate set-up the same can be proven for any *distribution* (generalized function) on the boundary.

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[1] Up to scalar multiples, the Laplacian is the unique second-order differential operator on $\mathbb{R}^n$ that is translation-invariant, rotation-invariant, and annihilates constants. In fact, *every* translation-invariant and rotation-invariant differential operator on $\mathbb{R}^n$ is a *polynomial* in the Laplacian. In some natural non-Euclidean spaces there are motion-invariant differential operators *not* expressible in terms of the corresponding Laplacian.

1. *Mean-value property*

Thus, among other features, in two dimensions harmonic functions form a useful, strictly larger class of functions including holomorphic functions. For example, harmonic functions still enjoy a *mean-value* property, as holomorphic functions do:

[1.1] Theorem: *(Mean-value property)* For harmonic u on a neighborhood of the closed unit disk,

$$u(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta$$

Proof: Consider the rotation-averaged function

$$v(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}\cdot z)\,d\theta \qquad\qquad (\text{for } |z|\le 1)$$

Since the Laplacian Δ is *rotation-invariant*, v is a rotation-invariant *harmonic* function. In polar coordinates, for rotation-invariant functions $v(z) = f(|z|)$, the Laplacian is

$$\Delta v \;=\; \Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)f(\sqrt{x^2+y^2}) \;=\; \frac{\partial}{\partial x}\big(\frac{x}{|z|}f'(|z|)\big)+\frac{\partial}{\partial y}\big(\frac{y}{|z|}f'(|z|)\big)$$

$$=\; \frac{1}{|z|}f' - \frac{x^2}{|z|^3}f' + \frac{x^2}{|z|^2}f'' + \frac{1}{|z|}f' - \frac{y^2}{|z|^3}f' + \frac{y^2}{|z|^2}f'' \;=\; f'' + \frac{1}{|z|}f'$$

The ordinary differential equation $f'' + f'/r = 0$ on an interval $(0, R)$ is an equation of *Euler type*, meaning expressible in the form $r^2f'' + Brf' + Cf = 0$ with constants B, C. In general, such equations are solved by letting $f(r) = r^\lambda$, substituting, dividing through by r^λ, and solving the resulting *indicial equation* for λ:

$$\lambda(\lambda-1) + A\lambda + B \;=\; 0$$

Distinct roots λ_1, λ_2 of the indicial equation produce linearly independent solutions r^{λ_1} and r^{λ_2}. However, as in the case at hand, a repeated root λ produces a second solution $r^\lambda \cdot \log r$. Here, the indicial equation is $\lambda^2 = 0$, so the general solution is $a + b\log r$. When $b \neq 0$, the solution $a + b\log r$ blows up as $r \to 0^+$. Since $f(0) = v(0) = u(0)$ is finite, it must be that $b = 0$. Thus, a *rotation-invariant* harmonic function on the disk is *constant*. Thus, its average over a circle is its central value, proving the mean-value property for harmonic functions. ///

[1.2] Remark: One might worry about commutation of the Laplacian with the integration above. In the first place, it is clear that we *must* have this commutativity. Second, the best and most final argument for such is in terms of *Gelfand-Pettis* (also called *weak*) integrals of function-valued functions, rather than temporary elementary arguments.

[1.3] Remark: The solutions $a + b\log r$ do indeed exhaust the possible solutions: given $f'' + f'/r = 0$ on $(0, R)$, we see $r \cdot f'$ is *constant* because

$$\frac{\partial}{\partial r}\big(r\cdot f'\big) \;=\; r\cdot f'' + f' \;=\; r\cdot\big(-f'/r\big) + f' \;=\; 0$$

The class of harmonic functions includes useful non-holomorphic real-valued functions. For example, (real-valued) *logarithms of absolute values of non-vanishing holomorphic functions are harmonic:*

$$\log|f(z)| \;=\; \tfrac{1}{2}\cdot\Big(\log f + \log\overline{f}\Big) \;=\; \tfrac{1}{2}\cdot\Big(\text{holomorphic} + \text{anti-holomorphic}\Big)$$

so is annihilated by $\Delta = 4\frac{\partial}{\partial z} \circ \frac{\partial}{\partial \overline{z}}$.

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2. *Poisson's formula and kernel for the disk*

The mean-value property will yield

[2.1] Corollary: *(Poisson's formula)* For u harmonic on a neighborhood of the closed unit disk $|z| \le 1$, u is expressible in terms of its boundary values on $|z| = 1$ by

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1-|z|^2}{|z-e^{i\theta}|^2}\, d\theta \qquad\qquad (\text{for } |z| < 1)$$

Up to scalars, the *Poisson kernel function* is

$$P(e^{i\theta}, z) \;=\; \frac{1-|z|^2}{|z-e^{i\theta}|^2}$$

Proof: Pre-composition $h \circ f$ of a harmonic function h with a holomorphic function f yields a harmonic function. With φ_z the linear fractional transformation given by matrix $\varphi_z \sim \begin{pmatrix} 1 & z \\ \overline{z} & 1 \end{pmatrix}$, the mean-value property for $u \circ \varphi_z$ gives

$$u(z) \;=\; (u \circ \varphi_z)(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} (u \circ \varphi_z)(e^{i\theta})\, d\theta$$

Linear fractional transformations stabilizing the unit disk map the unit circle to itself. Replace $e^{i\theta}$ by $e^{i\theta'} = \varphi_z^{-1}(e^{i\theta})$

$$u(z) \;=\; (u \circ \varphi_z)(0) \;=\; \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta'$$

Computing the change of measure will yield the Poisson formula. This is computed by

$$ie^{i\theta'} \cdot \frac{\partial \theta'}{\partial \theta} \;=\; \frac{\partial}{\partial \theta} e^{i\theta'} \;=\; \frac{ie^{i\theta}}{1-\overline{z}e^{i\theta}} + \frac{i\overline{z}e^{i\theta}(e^{i\theta}-z)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{ie^{i\theta} - i\overline{z}e^{2i\theta} + i\overline{z}e^{2i\theta} - ie^{i\theta}|z|^2}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{ie^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2}$$

Thus,

$$\frac{\partial \theta'}{\partial \theta} \;=\; \frac{1}{e^{i\theta'}} \frac{ie^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{1-\overline{z}e^{i\theta}}{e^{i\theta}-z} \cdot \frac{e^{i\theta}(1-|z|^2)}{(1-\overline{z}e^{i\theta})^2} \;=\; \frac{1-|z|^2}{|z-e^{i\theta}|^2}$$

giving the asserted integral. ///

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3. *Dirichlet problem on the disk*

As a sort of converse to the existence of the Poisson formula and Poisson kernel, the *Dirichlet problem* on the closed unit disk in $\mathbb{C} \approx \mathbb{R}^2$ is posed by specifying of a continuous function f on the circle $S^1 = \{z : |z| = 1\}$, and asking for harmonic u on $|z| < 1$, extending to a continuous function on $|z| \le 1$, such that $u|_{S^1} = f$. That is, the following asserts that the collection of harmonic functions is a correct collection of functions in which to pose the problem of reconstituting the interior values of a function from its boundary values, if we are to have existence and uniqueness. In contrast, the collection of holomorphic functions on a disk is too small, since the boundary values have Fourier series with no negative terms (see below). On another hand, any class of functions strictly larger than harmonic functions will include non-zero functions with boundary values identically 0.

[3.1] Corollary: Given a continuous function f on the circle $S^1 = \{z : |z| = 1\}$, there is a unique harmonic function u on the open unit disk extending to a continuous function on the closed unit disk and $u|_{S^1} = f$. In particular,

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\cdot\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \qquad\qquad (\text{for } |z|<1)$$

For real-valued f, the function u is also real-valued.

Proof: This proof re-discovers the Poisson kernel.

To skirt some secondary analytical complications, suppose that f is at least C^1, so that the partial sums of the Fourier series of f converge uniformly and absolutely pointwise to f, so

$$f(e^{it}) \;=\; \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e^{int} \;=\; \lim_{N\to\infty}\sum_{|n|\le N}\widehat{f}(n)\,e^{int}$$

where

$$\widehat{f}(n) \;=\; \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,f(e^{i\theta})\,d\theta$$

Note that on $|z| = 1$ with $z = e^{it}$ we can write

$$e^{int} \;=\; \begin{cases} z^n & (\text{for } n\ge 0 \text{ and } z=e^{it})\\[2ex] \overline{z}^{|n|} & (\text{for } n<0 \text{ and } z=e^{it})\end{cases}$$

In these terms,

$$\sum_{|n|\le N}\widehat{f}(n)\,e^{int} \;=\; \sum_{|n|\le N}\Big(\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\,f(e^{i\theta})\,d\theta\Big)\cdot e^{int} \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\Big(\sum_{|n|\le N} e^{-in\theta}\cdot e^{int}\Big)\,d\theta$$

$$=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\Big(\frac{1-(ze^{-i\theta})^{N+1}}{1-ze^{-i\theta}} + \frac{\overline{z}e^{i\theta}-(\overline{z}e^{i\theta})^{N+1}}{1-\overline{z}e^{i\theta}}\Big)\,d\theta$$

After moving z to $|z|<1$, we can take the limit $N\to\infty$, and

$$\sum_{n\ge 0}\widehat{f}(n)\,z^n + \sum_{n<0}\widehat{f}(n)\,\overline{z}^{|n|} \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\Big(\frac{1}{1-ze^{-i\theta}} + \frac{\overline{z}e^{i\theta}}{1-\overline{z}e^{i\theta}}\Big)\,d\theta$$

The left-hand side is the sum of an anti-holomorphic and a holomorphic function, so is harmonic. Simplifying slightly,

$$\frac{1}{1-ze^{-i\theta}} + \frac{\overline{z}e^{i\theta}}{1-\overline{z}e^{i\theta}} \;=\; \frac{1-\overline{z}e^{i\theta}+\overline{z}e^{i\theta}-|z|^2}{|1-ze^{-i\theta}|^2} \;=\; \frac{1-|z|^2}{|1-ze^{-i\theta}|^2} \;=\; \frac{1-|z|^2}{|e^{i\theta}-z|^2} \;=\; \frac{1-|z|^2}{|z-e^{i\theta}|^2}$$

Thus,

$$\sum_{n\ge 0}\widehat{f}(n)\,z^n + \sum_{n<0}\widehat{f}(n)\,\overline{z}^{|n|} \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta$$

To see that we recover the original boundary values, let $z = re^{it}$ with $r>0$, so

$$\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta \;=\; \lim_{r\to 1}\Big(\sum_{n\ge 0}\widehat{f}(n)\,r^n\,e^{int} + \sum_{n<0}\widehat{f}(n)\,r^{|n|}\,e^{int}\Big) \;=\; \sum_n \widehat{f}(n)e^{int} \;=\; f(e^{it})$$

using the good convergence of the Fourier series for f on the circle. ///

4. *Poisson kernel for upper half-plane*

Again using the fact that $h \circ f$ is harmonic for h harmonic and f holomorphic, we can transport the Poisson kernel $P(e^{i\theta}, z)$ for the disk to a Poisson kernel for the upper half-plane $\mathfrak{H}$ via the Cayley map $C : z \to (z+i)/(iz+1)$. The Cayley map gives a holomorphic isomorphism of the disk to the upper half-plane, and of the circle (with i removed) to the real line.

In the relation

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\, P(e^{i\theta}, z)\, d\theta$$

replace z by $C^{-1}z$ with $z \in \mathfrak{H}$, and replace $e^{i\theta}$ by $C^{-1}(z) = (t-i)/(-it+1)$ with $t \in \mathbb{R}$: using $d\theta = d(e^{i\theta})/ie^{i\theta}$,

$$(u \circ C^{-1})(z) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} (f \circ C^{-1})(t)\, P(C^{-1}(t), C^{-1}z)\, \frac{d\left(\frac{t-i}{-it+1}\right)}{i\frac{t-i}{-it+1}}$$

The change of measure can be determined by the natural computation

$$\frac{(t+\delta)-i}{-i(t+\delta)+1} \;=\; \frac{(t+\delta)-i}{-it+1-i\delta} \;=\; \frac{(t+\delta)-i}{(-it+1)(1-i\delta(1-it)^{-1})} \;=\; \frac{(t+\delta)-i}{-it+1}(1-i\delta(1-it)^{-1})^{-1}$$

$$= \Big(\frac{t-i}{-it+1} + \frac{\delta}{-it+1}\Big)\Big(1 + i\delta(1-it)^{-1} + O(\delta^2)\Big) \;=\; \frac{t-i}{-it+1} + \delta\cdot\Big(\frac{1}{-it+1} + \frac{t-i}{-it+1}\frac{i}{-it+1}\Big) + O(\delta^2)$$

$$= \frac{t-i}{-it+1} + \delta\cdot\frac{2}{(-it+1)^2} + O(\delta^2)$$

Thus,

$$\frac{d\left(\frac{t-i}{-it+1}\right)}{i\frac{t-i}{-it+1}} \;=\; dt\cdot\frac{\frac{2}{(-it+1)^2}}{i\frac{t-i}{-it+1}} \;=\; dt\cdot\frac{2}{(1+it)(1-it)} \;=\; \frac{2\,dt}{1+t^2}$$

With $z = x+iy$ in $\mathfrak{H}$, the Poisson kernel itself becomes

$$P_{\mathfrak{H}}(t, x+iy) \;=\; P_{\text{disk}}(C^{-1}(t), C^{-1}(z)) \;=\; \frac{1-|C^{-1}z|^2}{|C^{-1}z - C^{-1}t|^2} \;=\; \frac{1-\left|\frac{z-i}{-iz+1}\right|^2}{\left|\frac{z-i}{-iz+1} - \frac{t-i}{-it+1}\right|^2}$$

$$= \frac{|1-it|^2\cdot(|1-iz|^2 - |1+iz|^2)}{\big|(z-i)(1-it) - (1-iz)(t-i)\big|^2} \;=\; \frac{|1-it|^2\cdot\big(\big(x^2+(y+1)^2\big) - \big(x^2+(y-1)^2\big)\big)}{\big|(z-i-itz-t) - (t-itz-i-z)\big|^2}$$

$$= \frac{|1-it|^2\cdot 4y}{4|z-t|^2} \;=\; \frac{|1-it|^2\cdot y}{(x-t)^2+y^2}$$

Thus, with all these coordinate changes,

$$(u\circ C^{-1})(z) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}} (f\circ C^{-1})(t)\,\frac{y}{(x-t)^2+y^2}\,2\,dt \;=\; \frac{1}{\pi}\int_{\mathbb{R}} (f\circ C^{-1})(t)\,\frac{y}{(x-t)^2+y^2}\,dt$$

That is, replacing $f\circ C^{-1}$ by f and $u\circ C^{-1}$ by u, for suitable function f on the real line, a harmonic function u on the upper half-plane with boundary value f on $\mathbb{R}$ is given by

$$u(x+iy) \;=\; \frac{1}{\pi}\int_{\mathbb{R}} f(t)\,\frac{y}{(x-t)^2+y^2}\,dt \qquad \text{(Poisson formula on } \mathfrak{H}\text{)}$$