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## 12. Weierstrass and Hadamard products

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1. Weierstrass products
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Apart from factorization of polynomials, after Euler's

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

there is Euler's product for  $\Gamma(z)$ , which he used as the *definition* of the Gamma function:

$$\int_0^{\infty} e^{-t} t^z \frac{dt}{t} = \Gamma(z) = \frac{1}{z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}}$$

where the Euler-Mascheroni constant  $\gamma$  is essentially defined by this relation. The integral (Euler's) converges for  $\operatorname{Re}(z) > 0$ , while the product (Weierstrass') converges for all complex  $z$  except non-positive integers. Because the exponential factors are *linear*, and can *cancel*,

$$\frac{1}{\Gamma(z) \cdot \Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \cdot \sin \pi z$$

*Linear* exponential factors are exploited in Riemann's *explicit formula* [Riemann 1859], derived from equality of the *Euler product* and *Hadamard product* [Hadamard 1893] for the zeta function  $\zeta(s) = \sum_n \frac{1}{n^s}$  for  $\operatorname{Re}(s) > 1$ :

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s) = \frac{e^{a+bs}}{s-1} \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right) e^{-s/2n}$$

where the product expansion of  $\Gamma(\frac{s}{2})$  is visible, corresponding to *trivial zeros* of  $\zeta(s)$  at negative even integers, and  $\rho$  ranges over all other, *non-trivial* zeros, known to be in the *critical strip*  $0 < \operatorname{Re}(s) < 1$ .

The hard part of the proof (below) of Hadamard's theorem is essentially that of [Ahlfors 1953/1966], with various rearrangements. A somewhat different argument is in [Lang 1993]. Some standard folkloric proofs of supporting facts about harmonic functions are recalled.

## 1. Weierstrass products

Given a sequence of complex numbers  $z_j$  with no accumulation point in  $\mathbb{C}$ , we will construct an entire function with zeros exactly the  $z_j$ .

### [1.1] Basic construction

Taylor-MacLaurin polynomials of  $-\log(1-z)$  will play a role: let

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots + \frac{z^n}{n}$$

We will exhibit a sequence of integers  $n_j$  giving an *absolutely convergent infinite product* vanishing precisely at the  $z_j$ , with vanishing at  $z=0$  accommodated by a suitable leading factor  $z^m$ , of the form

$$z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_{n_j}(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^{n_j}}{n_j z_j^{n_j}}\right)$$

Absolute convergence of  $\sum_j \log(1+a_j)$  implies absolute convergence of the infinite product  $\prod_j(1+a_j)$ . Thus, we show that

$$\sum_j \left| \log\left(1 - \frac{z}{z_j}\right) + p_{n_j}\left(\frac{z}{z_j}\right) \right| < \infty$$

Fix a large radius  $R$ , keep  $|z| < R$ , and ignore the finitely-many  $z_j$  with  $|z_j| < 2R$ , so in the following  $|z/z_j| < \frac{1}{2}$ . Using the power series expansion of  $\log$ ,

$$\left| \log\left(1 - \frac{z}{z_j}\right) - p_n\left(\frac{z}{z_j}\right) \right| \leq \frac{1}{n+1} \cdot \left|\frac{z}{z_j}\right|^{n+1} + \frac{1}{n+2} \cdot \left|\frac{z}{z_j}\right|^{n+2} + \dots \leq \frac{1}{n+1} \cdot \frac{|z/z_j|^{n+1}}{1-|z/z_j|} \leq 2 \cdot \frac{|z/z_j|^{n+1}}{n+1}$$

Thus, we want a sequence of positive integers  $n_j$  such that

$$\sum_{|z_j| \geq 2R} \frac{|z/z_j|^{n_j+1}}{n_j+1} < \infty \quad (\text{with } |z| < R)$$

The choice of  $n_j$ 's must be compatible with enlarging  $R$ , and this is easily arranged. For example,  $n_j = j-1$  succeeds:

$$\sum_j \left|\frac{z}{z_j}\right|^j = \sum_{|z_j| < 2R} \left|\frac{z}{z_j}\right|^j + \sum_{|z_j| \geq 2R} \left|\frac{z}{z_j}\right|^j \leq \sum_{|z_j| < 2R} \left|\frac{z}{z_j}\right|^j + \sum_j 2^{-j}$$

Since  $\{z_j\}$  is discrete, the sum over  $|z_j| < 2R$  is finite, giving convergence, and convergence of the infinite product with  $n_j = j$ :

$$\prod_j \left(1 - \frac{z}{z_j}\right) e^{p_j(z/z_j)} = \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^j}{jz_j^j}\right)$$

### [1.2] Canonical products and genus

Given entire  $f$  with zeros  $z_j \neq 0$  and a zero of order  $m$  at 0, ratios

$$\varphi(z) = \frac{f(z)}{z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)}}$$

with *convergent* infinite products are *entire*, and *do not vanish*. Non-vanishing entire  $\varphi$  has an entire *logarithm*:

$$g(z) = \log \varphi(z) = \int_0^z \frac{\varphi'(\zeta) d\zeta}{\varphi(\zeta)}$$

Thus, non-vanishing entire  $\varphi$  is expressible as

$$\varphi(z) = e^{g(z)} \quad (\text{with } g \text{ entire})$$

Thus, the most general entire function with prescribed zeros is of the form

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)} \quad (\text{with } g \text{ entire})$$

With fixed  $f$ , altering the  $n_j$  necessitates a corresponding alteration in  $g$ .

We are most interested in zeros  $\{z_j\}$  allowing a *uniform* integer  $h$  giving convergence of the infinite product in an expression

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

When  $f$  admits such a product expression with a uniform  $h$ , a product expression with *minimal* uniform  $h$  is a *canonical product* for  $f$ .

When, further, the leading factor  $e^{g(z)}$  for  $f$  has  $g(z)$  a *polynomial*, the *genus* of  $f$  is the maximum of  $h$  and the *degree* of  $g$ .

## 2. Poisson-Jensen formula

Jensen's formula and the Poisson-Jensen formula are essential in the difficult half of the Hadamard theorem (below) comparing *genus* of an entire function to its *order of growth*.

The logarithm  $u(z) = \log |f(z)|$  of the absolute value  $|f(z)|$  of a non-vanishing holomorphic function  $f$  on a neighborhood of the unit disk is *harmonic*, that is, is annihilated by  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ : expand

$$\Delta \log |f(z)| = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \left(\frac{1}{2} \log f(z) + \frac{1}{2} \log \bar{f}(z)\right)$$

Conveniently, the two-dimensional Laplacian is the product of the Cauchy-Riemann operator and its conjugate. Since  $\log f$  is holomorphic and  $\log \bar{f}$  is *anti-holomorphic*, both are annihilated by the product of the two linear operators, so  $\log |f(z)|$  is harmonic.

Thus,  $\log |f(z)|$  satisfies the *mean-value property* for harmonic functions

$$\log |f(0)| = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

Next, let  $f$  have zeros  $\rho_j$  in  $|z| < 1$  but none on the unit circle. We manufacture a holomorphic function  $F$  from  $f$  but without zeros in  $|z| < 1$ , and with  $|F| = |f|$  on  $|z| = 1$ , by the standard ruse

$$F(z) = f(z) \cdot \prod_j \frac{1 - \bar{\rho}_j z}{z - \rho_j}$$

Indeed, for  $|z| = 1$ , the numerator of each factor has the same absolute value as the denominator:

$$|z - \rho_j| = \left| \frac{1}{z} - \bar{\rho}_j \right| = \frac{1}{|z|} \cdot |1 - \bar{\rho}_j z| = |1 - \bar{\rho}_j z|$$

For simplicity, suppose no  $\rho_j$  is 0. Applying the mean-value identity to  $\log |F(z)|$  gives

$$\log |f(0)| - \sum_j \log |\rho_j| = \log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

and then the basic *Jensen's formula*

$$\log |f(0)| - \sum_j \log |\rho_j| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta \quad (\text{for } |\rho_j| < 1)$$

The Poisson-Jensen formula is obtained by replacing 0 by an arbitrary point  $z$  inside the unit disk, by replacing  $f$  by  $f \circ \varphi_z$ , where  $\varphi_z$  is a *linear fractional transformation* mapping  $0 \rightarrow z$  and stabilizing<sup>[1]</sup> the unit disk:

$$\varphi_z = \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} : w \rightarrow \frac{w + z}{\bar{z}w + 1}$$

This replaces the zeros  $\rho_j$  by  $\varphi_z^{-1}(\rho_j) = \frac{\rho_j - z}{-\bar{z}\rho_j + 1}$ . Instead of the mean-value property expressing  $f(0)$  as an integral over the circle, use the *Poisson formula* for  $f(z)$ . This gives the basic *Poisson-Jensen formula*

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{z}\rho_j + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < 1, |\rho_j| < 1)$$

Generally, for holomorphic  $f$  on a neighborhood of a disk of radius  $r > 0$  with zeros  $\rho_j$  in that disk, apply the previous to  $f(r \cdot z)$  with zeros  $\rho_j/r$  in the unit disk:

$$\log |f(r \cdot z)| - \sum_j \log \left| \frac{\rho_j/r - z}{-\bar{z}\rho_j/r + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r \cdot e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < 1)$$

Replacing  $z$  by  $z/r$  gives

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j/r - z/r}{-\bar{z}\rho_j/r^2 + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{1 - |z/r|^2}{|z/r - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < r)$$

which rearranges to the general *Poisson-Jensen formula*

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{z}\rho_j/r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta \quad (\text{for } |z| < r, |\rho_j| < r)$$

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[1] To verify that such maps stabilize the unit disk, expand the natural expression:

$$\begin{aligned} 1 - \left| \frac{w + z}{\bar{z}w + 1} \right|^2 &= |\bar{z}w + 1|^{-2} \cdot \left( |\bar{z}w + 1|^2 - |w + z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot \left( |zw|^2 + \bar{z}w + z\bar{w} + 1 - |w|^2 - \bar{z}w - z\bar{w} - |z|^2 \right) \\ &= |\bar{z}w + 1|^{-2} \cdot \left( |zw|^2 + 1 - |w|^2 - |z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot (1 - |z|^2) \cdot (1 - |w|^2) > 0 \end{aligned}$$

The case  $z = 0$  is the general *Jensen formula* for arbitrary radius  $r$ :

$$\log |f(0)| - \sum_j \log \left| \frac{\rho_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad (\text{with } |\rho_j| < r)$$

### 3. Hadamard products

The *order* of an entire function  $f$  is the smallest positive real  $\lambda$ , if it exists, such that, for every  $\varepsilon > 0$ ,

$$|f(z)| \leq e^{|z|^{\lambda+\varepsilon}} \quad (\text{for all sufficiently large } |z|)$$

Recall that, in an infinite product expression with compensating exponential factors with *uniform* degree  $h$

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

when the exponent  $g(z)$  is *polynomial*, the *genus* of  $f$  is the maximum of  $h$  and the *degree* of  $g$ .

**[3.1] Theorem:** (*Hadamard*) The *genus*  $h$  and order  $\lambda$  are related by  $h \leq \lambda < h + 1$ . In particular, one is *finite* if and only the other is finite.

*Proof:* First, the easier half. For  $f$  of finite genus  $h$  expressed as

$$f(z) = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

the leading exponent is polynomial  $g$  of degree at most  $h$ , so  $e^{g(z)}$  is of *order* at most  $h$ . The order of a product is at most the maximum of the orders of the factors, so it suffices to prove that the order of the infinite product is at most  $h + 1$ .

The assumption that  $h$  is the genus of  $f$  is equivalent to

$$\sum_j \frac{1}{|z_j|^{h+1}} < \infty$$

We use this to directly estimate the infinite product and show that it has order of growth  $\lambda < h + 1$ .

Toward an estimate on  $F_h(w) = (1 - w)e^{p_h(w)}$  applicable for *all*  $w$ , not merely for  $|w| < 1$ , we collect some inequalities. There is the basic

$$\log |F_h(w)| = \log |(1 - w)e^{p_{h-1}(w)} \cdot e^{w^h/h}| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \quad (\text{for all } w)$$

As before, for  $|w| < 1$ ,

$$\log |F_h(w)| \leq \frac{1}{h+1} \cdot |w|^{h+1} + \frac{1}{h+2} \cdot |w|^{h+2} + \dots \leq |w|^{h+1} \cdot \frac{1}{1-|w|} \quad (\text{for } |w| < 1)$$

This gives  $(1 - |w|) \cdot \log |F_h(w)| \leq |w|^{h+1}$  for  $|w| < 1$ . Adding to the latter the basic relation multiplied by  $|w|$  gives

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } |w| < 1)$$

In fact, the latter inequality also holds for  $|w| \geq 1$  and  $\log |F_{h-1}(w)| \geq 0$ , from the basic relation. For  $\log |F_{h-1}(w)| < 0$  and  $|w| \geq 1$ , from the basic relation,

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \leq \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } \log |F_{h-1}(w)| < 0 \text{ and } |w| \geq 1)$$

Now prove  $\log |F_h(w)| \ll_h |w|^{h+1}$ , by induction on  $h$ . For  $h = 0$ , from  $\log |x| \leq |x| - 1$ ,

$$\log |1 - w| \leq |1 - w| - 1 \leq 1 + |w| - 1 = |w|$$

Assume  $\log |F_{h-1}(w)| \ll_h |w|^h$ . For  $|w| < 1$ , we reach the desired conclusion by

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + \left(1 + \frac{1}{h}\right) |w|^{h+1} \ll_h |w| \cdot |w|^h + \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } |w| < 1)$$

For  $|w| \geq 1$  and  $\log |F_{h-1}(w)| > 0$ , from the basic relation

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h + \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| > 0)$$

For  $\log |F_{h-1}(w)| \leq 0$  and  $|w| \geq 1$ , from the basic relation we already have

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| < 0)$$

This proves  $\log |F_h(w)| \ll_h |w|^{h+1}$  for all  $w$ .

Estimate the infinite product:

$$\log \left| \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_h(z/z_j)} \right| = \sum_j \log \left| \left(1 - \frac{z}{z_j}\right) \cdot e^{p_h(z/z_j)} \right| \ll_h \sum_j \left| \frac{z}{z_j} \right|^{h+1} < \infty$$

since  $\sum 1/|z_j|^{h+1}$  converges. Thus, such an infinite product has *growth order*  $\lambda \leq h + 1$ .

Now the difficult half of the proof. Let  $h \leq \lambda < h + 1$ . *Jensen's formula* will show that the zeros  $z_j$  are sufficiently spread out for convergence of  $\sum 1/|z_j|^{h+1}$ . Without loss of generality, suppose  $f(0) \neq 0$ . From

$$\log |f(0)| - \sum_j \log \left| \frac{z_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad (\text{with } |z_j| < r)$$

certainly

$$\sum_{|z_j| < r/2} \log 2 \leq \sum_{|\rho_j| < r/2} -\log \left| \frac{\rho_j}{r} \right| \ll_\varepsilon -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} r^{\lambda+\varepsilon} d\theta \ll r^{\lambda+\varepsilon} \quad (\text{for every } \varepsilon > 0)$$

With  $\nu(r)$  the number of zeros inside the disk of radius  $r$ , this gives

$$\lim_{r \rightarrow +\infty} \frac{\nu(r)}{r^{\lambda+\varepsilon}} = 0 \quad (\text{for all } \varepsilon > 0)$$

Order the zeros by absolute value:  $|z_1| \leq |z_2| \leq \dots$  and for simplicity suppose no two have the same size. Then  $j = \nu(|z_j|) \ll_\varepsilon |z_j|^{\lambda+\varepsilon}$ . Thus,

$$\sum \frac{1}{|z_j|^{h+1}} \ll_\varepsilon \sum \frac{1}{(j^{\frac{1}{\lambda+\varepsilon}})^{h+1}} = \sum \frac{1}{j^{\frac{h+1}{\lambda+\varepsilon}}}$$

The latter converges for  $\frac{h+1}{\lambda+\varepsilon} > 1$ , that is, for  $\lambda + \varepsilon < h + 1$ . When  $\lambda < h + 1$ , there is  $\varepsilon > 0$  making such an equality hold.

It remains to show that the entire function  $g(z)$  in the leading exponential factor is of degree at most  $h + 1$ , by showing that its  $(h + 1)^{th}$  derivative is 0.

In the Poisson-Jensen formula

$$\log |f(z)| - \sum_{|z_j| < r} \log \left| \frac{z_j - z}{-\bar{z}_j z_j / r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta \quad (\text{for } |z| < r)$$

application of  $\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$  annihilates the anti-holomorphic parts, returning us to an equality of holomorphic functions, as follows. The effect on the integrand is

$$\begin{aligned} 2 \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} &= 2 \frac{-\bar{z}}{(z - re^{i\theta})(\bar{z} - re^{-i\theta})} - \frac{r^2 - |z|^2}{(z - re^{i\theta})^2 (\bar{z} - re^{-i\theta})} \\ &= 2 \frac{-|z|^2 + \bar{z}re^{i\theta} - r^2 + |z|^2}{(z - re^{i\theta})^2 (\bar{z} - re^{-i\theta})} = 2 \frac{re^{i\theta}}{(re^{i\theta} - z)^2} \end{aligned}$$

Thus,

$$\frac{f'(z)}{f(z)} - \sum_{|z_j| < r} \frac{1}{z - z_j} + \sum_{|z_j| < r} \frac{\bar{z}_j}{\bar{z}_j z - r^2} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^2} d\theta$$

Further differentiation  $h$  times in  $z$  gives

$$\left( \frac{f'(z)}{f(z)} \right)^{(h)} = \sum_{|z_j| < r} \frac{(-1)^h h!}{(z - z_j)^{h+1}} - \sum_{|z_j| < r} \frac{(-1)^h h! \cdot \bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} + \frac{(-1)^h (h+1)!}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta$$

We claim that the second sum and the integral go to 0 as  $r \rightarrow +\infty$ .

Regarding the integral, Cauchy's integral formula gives

$$\int_0^{2\pi} \frac{re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta = 0$$

Letting  $M_r$  be the maximum of  $|f|$  on the circle of radius  $r$ , taking  $|z| < r/2$ , up to sign the integral is

$$\int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) \cdot \frac{2re^{i\theta} d\theta}{(z - re^{i\theta})^{h+2}} \ll \frac{1}{r^{h+1}} \int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) d\theta \ll_\varepsilon \frac{r^{\lambda+\varepsilon}}{r^{h+1}} \cdot \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta$$

Jensen's formula gives

$$\frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta \leq -\log |f(0)|$$

Thus, for  $\lambda + \varepsilon < h + 1$  the integral goes to 0 as  $r \rightarrow +\infty$ .

For the second sum, again take  $|z| < r/2$ , so for  $|z_j| < r$

$$\left| \frac{\bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} \right| \leq \frac{|z_j^{h+1}|}{(r^2 - |z_j| \cdot \frac{r}{2})^{h+1}} \ll \frac{|z_j^{h+1}|}{r^{h+1} (r - |z_j|)^{h+1}} \ll \frac{1}{r^{h+1}}$$

We already showed that the number  $\nu(r)$  of  $|z_j| < r$  satisfies  $\lim \nu(r)/r^{h+1} = 0$ . Thus, this sum goes to 0 as  $r \rightarrow +\infty$ . Taking the limit,

$$\left( \frac{f'}{f} \right)^{(h)} = (-1)^h h! \sum_j \frac{1}{(z - z_j)^{h+1}}$$

Returning to  $f(z) = e^{g(z)} \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_h(z/z_j)}$ , taking logarithmic derivative gives

$$\frac{f'}{f} = g' + \sum_j \left( \frac{1}{z - z_j} + \frac{p'_h(z/z_j)}{z_j} \right)$$

and taking  $h$  further derivatives gives

$$\left( \frac{f'}{f} \right)^{(h)} = g^{(h+1)} + \sum_j \frac{(-1)^h h!}{(z - z_j)^{h+1}}$$

Since the  $h^{\text{th}}$  derivative of  $f'/f$  is the latter sum,  $g^{(h+1)} = 0$ , so  $g$  is a polynomial of degree at most  $h$ .  
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