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Uncertainty principles in Fourier analysis

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[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2020-21/12e_heisenberg_uncertainty.pdf]

The Heisenberg Uncertainty Principle is a *theorem* about Fourier transforms. ^[1]

For suitable f on \mathbb{R} ,

$$|f|_{L^2}^2 = \int_{\mathbb{R}} |f|^2 = - \int_{\mathbb{R}} x(f \cdot \bar{f})' = -2\operatorname{Re} \int_{\mathbb{R}} x f \bar{f}' \quad (\text{integrating by parts})$$

That is,

$$|f|_{L^2}^2 = ||f|_{L^2}^2| = \left| \int_{\mathbb{R}} |f|^2 \right| = \left| -2\operatorname{Re} \int_{\mathbb{R}} x f \bar{f}' \right| \leq 2 \int_{\mathbb{R}} |x f \bar{f}'|$$

Next,

$$2 \int_{\mathbb{R}} |x f \cdot \bar{f}'| \leq 2 \cdot |x f|_{L^2} \cdot |f'|_{L^2} \quad (\text{Cauchy-Schwarz-Bunyakovsky})$$

Since Fourier transform is an isometry, and since Fourier transform converts derivatives to multiplications,

$$|f'|_{L^2} = |\widehat{f'}|_{L^2} = 2\pi |\widehat{\xi f}|_{L^2}$$

Thus, we obtain the **Heisenberg inequality**

$$|f|_{L^2}^2 \leq 4\pi \cdot |x f|_{L^2} \cdot |\widehat{\xi f}|_{L^2}$$

More generally, a similar argument gives, for any $x_o \in \mathbb{R}$ and any $\xi_o \in \mathbb{R}$,

$$|f|_{L^2}^2 \leq 4\pi \cdot |(x - x_o)f|_{L^2} \cdot |(\xi - \xi_o)\widehat{f}|_{L^2}$$

Imagining that $f(x)$ is the probability that a particle's *position* is x , and $\widehat{f}(\xi)$ is the probability that its *momentum* is ξ , Heisenberg's inequality gives a lower bound on how *spread out* these two probability distributions must be. The physical assumption is that position and momentum *are* related by Fourier transform.

[1] I first saw Heisenberg's Uncertainty Principle presented directly as a theorem about Fourier transforms in Folland's 1983 Tata Lectures on PDE.