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13. Trigonometric functions, elliptic functions

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1. *Reconsideration of* $\sin x$

Although we already know quite a bit about trigonometric functions and their role in calculus, their treatment can be redone to emphasize parallels for *elliptic integrals* and *elliptic functions*.

[1.1] Integral defining $\arcsin x$ The length of arc of a piece of a circle is $x^2 + y^2 = 1$ is

$$\text{arc length of circle fragment from 0 to } x \;=\; \int_0^x \sqrt{1+y'^2}\,dt \;=\; \int_0^x \sqrt{1+\Big(\frac{d}{dt}\sqrt{1-t^2}\Big)^2}\,dt$$

$$= \int_0^x \sqrt{1+\Big(\frac{\frac{1}{2}\cdot(-2t)}{\sqrt{1-t^2}}\Big)^2}\,dt \;=\; \int_0^x \sqrt{1+\frac{t^2}{1-t^2}}\,dt \;=\; \int_0^x \frac{dt}{\sqrt{1-t^2}} \;=\; \arcsin x$$

[1.2] Periodicity The *periodicity* of $\sin x$ comes from the *multi-valuedness* of $\arcsin x$. The multi-valuedness is ascertainable from this integral, since the path from 0 to x can meander through the *complex plane*, going around the two points ± 1 special for the integral. The basic unit of this multi-valuedness is

$$\int_\gamma \frac{d\zeta}{\sqrt{1-\zeta^2}} \;=\; \pm 2\pi \qquad\qquad (\text{counter-clockwise circular path } \gamma \text{ enclosing both } \pm 1)$$

That is, a path integral from 0 to x could go from 0 to x along the real axis, but then add to the path a vertical line segment from x out to a circle of radius 2, traverse the circle an arbitrary number of times, come back along the same segment (thus cancelling the contribution from this segment).

There are two single-valued choices for $(1-z^2)^{-\frac{1}{2}}$ on any region complementary to an arc connecting ± 1. For example, it is easy to see a *Laurent expansion* in $|z| > 1$:

$$(1-z^2)^{-\frac{1}{2}} \;=\; \frac{\pm i}{z}\cdot\Big(1-\frac{1}{z^2}\Big)^{-\frac{1}{2}} \;=\; \frac{\pm i}{z}\cdot\Big(1-(-\tfrac{1}{2})\frac{1}{z^2}+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\big(\frac{1}{z^2}\big)^2+\ldots\Big) \;=\; \pm\Big(\frac{i}{z}+\frac{1/2}{z^3}+\frac{3/8}{z^5}+\ldots\Big)$$

Let γ be a path traversing counterclockwise a circle of radius more than 1 centered at 0. Integrals $\int_\gamma \frac{dz}{z^N}$ are 0 except for $N = 1$, in which case the integral is $2\pi i$. Thus,

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}} \;=\; \pm 2\pi$$

[1.3] Polynomial relation between $\sin x$ and $\sin' x$ Rewriting the integral as

$$\int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}} \;=\; x$$

and taking a derivative, by the fundamental theorem of calculus,

$$\sin' x \cdot \frac{1}{\sqrt{1-\sin^2}} \;=\; 1$$

Thus, $(\sin' x)^2 = 1 - (\sin x)^2$ and the algebraic relation between $\sin x$ and its derivative (we know it is $\cos x$) arises:

$$(\sin x)^2 + (\sin' x)^2 = 1$$

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2. *Construction of singly-periodic functions*

A *singly-periodic* function f on $\mathbb{C}$ is a (probably holomorphic or meromorphic) function such that for some $\omega \neq 0$

$$f(z+\omega) \;=\; f(z) \qquad\qquad \text{(for all } z \in \mathbb{C}\text{)}$$

or at least for z away from poles of f. Iterating the condition, for any integer n

$$f(z+n\omega) \;=\; f(z)$$

In other words, f is *invariant* under translation by the group $\mathbb{Z} \cdot \omega$ inside $\mathbb{C}$.

Feigning ignorance of the trigonometric (and exponential) function, whether as inverse functions to integrals or not, as a warm-up to the construction of *doubly*-periodic functions we should try to *construct* some singly-periodic functions.

[2.1] Construction and comparison to $\sin z$ For simplicity, take $\omega = 1$, and make holomorphic or meromorphic functions f such that

$$f(z+1) \;=\; f(z) \qquad\qquad \text{(for all } z \in \mathbb{C}\text{)}$$

That is, we want $\mathbb{Z}$-periodic functions on $\mathbb{C}$, hypothetically closely related to $\sin 2\pi z$. A fundamental approach to manufacturing such things is *averaging*, also called *periodicization* or *automorphizing*, as follows. For given function φ, consider

$$f(z) \;=\; \sum_{n\in\mathbb{Z}} \varphi(z+n)$$

If this converges nicely it is certainly periodic with period 1, since

$$f(z+1) \;=\; \sum_{n\in\mathbb{Z}} \varphi(z+1+n) \;=\; \sum_{n\in\mathbb{Z}} \varphi(z+n)$$

by replacing n by $n-1$, using the fact that we have summed over the *group* $\mathbb{Z}$, justifying rearrangement by absolute convergence.

An elementary function making the sum converge, apart from poles, is $\varphi(z) = 1/z^2$, so put

$$f(z) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2}$$

If we are lucky, this manufactures a function related to the *sine* function. Indeed, $f(z)$ has double poles at the zeros of $\sin \pi z$, so a plausible preliminary guess is that $f(z)$ is $1/(\sin \pi z)^2$.

To prove equality, perhaps after adjusting details, match poles, subtract, check that the difference goes to 0 as the imaginary part of z at infinity goes to ∞, and invoke Liouville. [1]

To do this, first determine the Laurent expansion of $f(z)$ near its poles. By periodicity, we'll understand all the poles if we understand the pole at $z = 0$. This is

$$f(z) \;=\; \frac{1}{z^2} + \sum_{n\neq 0} \frac{1}{(z+n)^2} \;=\; \frac{1}{z^2} + (\text{holomorphic near } z=0)$$

To understand the nature of each pole of $1/\sin^2 \pi z$, by periodicity it suffices to look near $z = 0$. Since

$$\sin \pi z \;=\; \pi z + (\pi z)^3/3! + \ldots \;=\; \pi z \cdot \big(1 + (\pi z)^2/3! + \ldots\big)$$

the inverse square is [2]

$$\frac{1}{\sin^2 \pi z} \;=\; \frac{1}{\pi z \cdot \big(1 + (\pi z)^2/3! + \ldots\big)} \;=\; \frac{1}{(\pi z)^2} \cdot \big(1 - (\pi z)^2/3! + \ldots\big) \;=\; \frac{1/\pi^2}{z^2} + \text{holomorphic at } 0$$

Correcting by π^2, the poles of $f(z)$ and $\pi^2/\sin^2 \pi z$ cancel:

$$f(z) - \frac{\pi^2}{\sin^2 \pi z} \;=\; \sum_n \frac{1}{(z+n)^2} - \frac{\pi^2}{\sin^2 \pi z} \;=\; \text{entire}$$

As $\text{Im}(z)$ becomes large, $f(z)$ goes to zero. For apparently different reasons, also

$$\frac{\pi^2}{\sin^2 \pi z} \;=\; \Big(\frac{2\pi i}{e^{\pi i z} - e^{-2\pi i z}}\Big)^2 \longrightarrow 0 \qquad\qquad (\text{as } |\text{Im}(z)| \to \infty)$$

Thus, by Liouville,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^2} \;=\; \frac{\pi^2}{\sin^2 \pi z}$$

[2.2] Differential equations for singly-periodic functions

The construction produced a *singly*-periodic function $\sum 1/(z+n)^2$ identifiable in terms of already-familiar items. The analogous discussion for *doubly*-periodic functions does *not* produce familiar objects. For practice, we now take another viewpoint that will also succeed with constructed *doubly*-periodic functions.

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[1] Liouville's theorem asserts that a *bounded entire function* is *constant*. As an immediate corollary, an entire function which is bounded *and* goes to 0 as the imaginary part of z goes to infinity is 0. Liouville's theorem is a striking instance of *rigidity*, where to prove two things equal, we need not prove something with infinite precision, but only demonstrate *sufficient* closeness to be able to infer equality. A trivial case of *rigidity* is that two integers are equal if they are within distance 1.

[2] The first two terms of the multiplicative inverse of a convergent power series $1 + c_1 z + c_2 z^2 + \ldots$ are easily determined, using $\frac{1}{1-r} = 1 + r + r^2 + \ldots$:

$$\frac{1}{1 + c_1 z + c_2 z^2 + \ldots} \;=\; \frac{1}{1 - (-c_1 z - c_2 z^2 - \ldots)}$$

$$=\; 1 + (-c_1 z - c_2 z^2 - \ldots) + (-c_1 z - c_2 z^2 - \ldots)^2 + \ldots \;=\; 1 - c_1 z + (\text{higher-order})$$

That is, with leading term 1, the coefficient of z changes simply by flipping sign.

The aim is to determine a differential equation satisfied by $f(z) = \sum 1/(z+n)^2$. In terms of Liouville, both f and its derivative

$$f'(z) \;=\; -2\sum \frac{1}{(z+n)^3}$$

go to 0 as $\mathrm{Im}(z) \to \pm\infty$, so if a polynomial $P(f, f')$ in f and f' cancels the poles, then $P(f, f')$ is necessarily *constant*, giving a polynomial relation [3] between f' and f. By periodicity, it suffices to consider the poles at $z = 0$, as before.

Noting that $f(-z) = f(z)$ assures vanishing of odd-order terms, let

$$f(z) \;=\; \frac{1}{z^2} + a + bz^2 + O(z^4) \qquad \text{so} \qquad f'(z) \;=\; \frac{-2}{z^3} + 2bz + O(z^3)$$

To cancel poles by a polynomial $P(f, f')$, the first step is to cancel the worst pole by $f^3 - (f'/-2)^2$: compute (carefully!?) that

$$f(z)^2 \;=\; \frac{1}{z^4} + \frac{2a}{z^2} + O(1) \qquad \text{and} \qquad f(z)^3 \;=\; \frac{1}{z^6} + \frac{3a}{z^4} + \frac{3a^2+3b}{z^2} + O(1)$$

and

$$\Big(\frac{f'(z)}{-2}\Big)^2 \;=\; \Big(\frac{1}{z^3} - bz + O(z^3)\Big)^2 \;=\; \frac{1}{z^6} - \frac{2b}{z^2} + O(1)$$

Thus,

$$\Big(\frac{f'(z)}{-2}\Big)^2 - f(z)^3 \;=\; -\frac{3a}{z^4} - \frac{3a^2+5b}{z^2} + O(1)$$

The $1/z^4$ term can be eliminated by adjusting by a multiple of $f(z)^2$:

$$\Big(\frac{f'(z)}{-2}\Big)^2 - f(z)^3 + 3a\cdot f(z)^2 \;=\; \frac{-3a^2 - 5b + 6a^2}{z^2} + O(1) \;=\; \frac{3a^2 - 5b}{z^2} + O(1)$$

Finally, subtract a multiple of $f(z)$ to eliminate the $1/z^2$ term:

$$\Big(\frac{f'(z)}{-2}\Big)^2 - f(z)^3 + 3a\cdot f(z)^2 - (3a^2 - 5b)\cdot f(z) \;=\; O(1)$$

In fact, since both f and f' go to zero as $\mathrm{Im}(z) \to \pm\infty$, the $O(1)$ term must be 0. Rearrangement produces a relation anticipating Weierstraß' analogous relation for constructed *doubly*-periodic functions:

$$\boxed{f'^2 \;=\; 4f^3 - 12af^2 + (12a^2 - 20b)f} \qquad \Big(\text{with } f(z) = \sum \frac{1}{(z+n)^2} = \frac{1}{z^2} + a + bz^2 + \ldots\Big)$$

Again, for other reasons, we know $f(z) = \pi^2/\sin^2 \pi z$.

[2.3] Remark: The existence of the relation made no use of higher Laurent coefficients, and at the same time explicitly demonstrates the dependence of the algebraic relation on the coefficients a, b in $f(z) = \frac{1}{z^2} + a + bz^2 + \ldots$. As discussed just below, in fact $a = 2\zeta(2) = \pi^2/3$ and $b = 6\zeta(4) = \pi^4/15$. Thus, miraculously, $12a^2 - 20b = 0$, and the relation is simply

$$\boxed{f'^2 \;=\; 4f^3 - 4\pi^2 f^2} \qquad \Big(\text{with } f(z) = \sum \frac{1}{(z+n)^2}\Big)$$

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[3] A non-linear polynomial relation between f and f' is a *non-linear*, probably hard-to-solve, differential equation. The difficulty of solving non-linear differential equations in general is not the point, however.

3. *Arc length of ellipses: elliptic integrals and elliptic functions*

One might naturally be interested in the integral for the length of a piece of arc of an ellipse. For example, the arc length of the piece the ellipse

$$x^2 + k^2y^2 \;=\; 1 \qquad\qquad \text{(with real } k \neq 0\text{)}$$

up to x, in the first quadrant, is

$$\int_0^x \sqrt{1+y'^2}\,dt \;=\; \int_0^x \sqrt{1+\frac{1}{k^2}\left(\frac{\frac{-1}{2}\cdot 2t}{\sqrt{1-t^2}}\right)^2}\,dt \;=\; \int_0^x \sqrt{1+\frac{1}{k^2}\frac{t^2}{1-t^2}}\,dt \;=\; \frac{1}{\sqrt{k}}\int_0^x \sqrt{\frac{k^2-(k^2-1)t^2}{1-t^2}}\,dt$$

For the *circle*, $k=1$, simplifying the numerator, and the value is as above:

$$\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \;=\; \arcsin x$$

Otherwise, there is no obvious reduction to elementary integrals. This leads to calling this integral an *elliptic integral*. [4] Many people studied the effect of changes of variables to transmute one form into another. [5] The immediate problems of computing arc length or evaluating integrals were eclipsed by the higher-level discovery by Abel and Jacobi (independently) in 1827 of the *double periodicity*[6] of functions $f(z)$ defined implicitly by

$$z \;=\; \int_0^{f(z)} \frac{d\zeta}{\sqrt{\text{quartic in } \zeta \text{ with distinct factors}}}$$

That is, there are *periods* ω_1 and ω_2 in $\mathbb{C}$ such that

$$f(z+\omega_1) \;=\; f(z+\omega_2) \;=\; f(z) \qquad\qquad \text{(for all } z\in\mathbb{C}\text{)}$$

with ω_1 and ω_2 linearly independent over $\mathbb{R}$. These ω_1 and ω_2 will arise as *integrals* of $1/\sqrt{\text{quartic}}$ over closed paths, which is why these integrals themselves have come to be called *period integrals*, or simply *periods*.

For example, consider

$$z = \int_0^{f(z)} \frac{d\zeta}{\sqrt{1+\zeta^4}}$$

[4] More generally, an *elliptic integral* is of the form

$$\int_a^b \frac{\text{(rational expression in } z\text{)}}{\sqrt{\text{cubic or quartic in } z}}\,dz$$

When the expression inside the radical has more than 4 zeros, or if the square root is replaced by a higher-order root, the integral's behavior is yet more complicated. The case of square root of cubic or quartic is the simplest beyond more elementary integrals. Abel and Jacobi and others *did* subsequently consider the more complicated cases, a popular pastime throughout the 19^{th} century.

[5] By 1757 Euler had studied relationships $dx/\sqrt{x^4+1} + dy/\sqrt{y^4+1} = 0$, leading to algebraic relations between x and y. Legendre (about 1811) studied transformations of such integrals, giving special reduced forms.

[6] Gauss later claimed he had found the double periodicity earlier, but had not published it. Abel and Jacobi published in 1827, and Legendre very civilly acknowledged their work in a new edition of his *Exercices de Calcul Intégral*. Later archival work did verify that Gauss had *privately* found the double periodicity in 1809.

The uniform *ambiguities* in the value of this integral viewed as a *path integral* from 0 to w are the values of the integrals along closed paths which circle an *even* number of the bad points $e^{2\pi ik/8}$ with $k = 1, 3, 5, 7$ (primitive 8^{th} roots of 1).

With a denominator decaying like $1/|\zeta|^2$ for large $|\zeta|$, the integral over large circles goes to 0 as the radius goes to $+\infty$. The integral around two points added to the integral around the other two points is equal to that outer circle integral, which is 0, so any two such integrals are merely negatives of each other.

Because of the decay for $|\zeta|$ large, the integral of $1/\sqrt{1+\zeta^4}$ along a path encircling $e^{2\pi i/8}$ and $e^{2\pi i\,3/8}$ is equal (via a deformation of the path) to the integral along the real axis, namely

$$\lambda \;=\; \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{1+t^4}}$$

Whatever else this may be, it is a positive real number. Similarly, the integral along a path encircling $e^{2\pi i/8}$ and $e^{2\pi i\,7/8}$ is equal (via a deformation of path) to the path integral along the imaginary axis, namely

$$\int_{-\infty}^{+\infty} \frac{d(it)}{\sqrt{1+(it)^4}} \;=\; i \cdot \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{1+t^4}} \;=\; i\lambda$$

since $i^4 = 1$. Thus, the double periodicity

$$f(z+\lambda) \;=\; f(z+i\lambda) \;=\; f(z)$$

The corresponding *period lattice* is

$$\Lambda \;=\; \mathbb{Z}\cdot\lambda + \mathbb{Z}\cdot i\lambda \subset \mathbb{C}$$

[3.1] Remark: In our example, the function $f(z)$ has *poles*. Notice that the integral along the positive real axis

$$\int_0^{+\infty} \frac{dt}{\sqrt{1+t^4}}$$

is absolutely convergent, with value $\lambda/2$, where λ is the *whole* integral on the real line, as just above. That is, $f(\lambda/2) = \infty$, which is to say that f has a pole at $\lambda/2$. Likewise, the integral along the upper imaginary axis is absolutely convergent, to $i\lambda/2$, so another pole is at $i\lambda/2$. And then the periodicity implies that there are poles (at least) at all points

$$\frac{\lambda}{2} + (m+ni)\lambda \qquad \frac{i\lambda}{2} + (m+ni)\lambda \qquad\qquad (\text{for } m, n \in \mathbb{Z})$$

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4. *Construction of doubly-periodic functions*

Once the existence of doubly-periodic functions is established via inverting elliptic integrals, the possibility of other constructions arises, just as we have alternative expressions for $\sin x$ in addition to $x = \int_0^{\sin x} dt/\sqrt{1-t^2}$.

A *lattice* in $\mathbb{C}$ is a subgroup of $\mathbb{C}$ of the form

$$\Lambda \;=\; \mathbb{Z}\cdot\omega_1 + \mathbb{Z}\cdot\omega_2 \qquad\qquad (\text{with } \omega_1, \omega_2 \text{ linearly independent over } \mathbb{R})$$

We want Λ-*periodic* functions, meaning meromorphic functions f on $\mathbb{C}$ such that

$$f(z+\lambda) \;=\; f(z) \qquad\qquad (\text{for all } z \in \mathbb{C} \text{ and } \lambda \in \Lambda)$$

These are *elliptic functions* with *period lattice* Λ. For a construction by *averaging*, just as we constructed $\pi^2/\sin^2 \pi z$, consider sums

$$\sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^k}$$

For $k > 2$, these are absolutely convergent and uniformly on compacta, and visibly invariant under $z \to z+\lambda$ for $\lambda \in \Lambda$. The smallest exponent for which this sum converges (for z not in Λ) is $k = 3$, but Weierstraß discovered[7] that it is best, to try to repair the convergence in the $k = 2$ case. The *Weierstraß $\wp$-function* is

$$\wp(z) \;=\; \wp_\Lambda(z) \;=\; \frac{1}{z^2} + \sum_{0\neq\lambda\in\Lambda} \left(\frac{1}{(z+\lambda)^2} - \frac{1}{\lambda^2}\right)$$

This *does* converge absolutely, but the argument for double periodicity is more complicated. Still, its derivative

$$\wp'(z) \;=\; \wp'_\Lambda(z) \;=\; -2 \sum_{\lambda\in\Lambda} \frac{1}{(z+\lambda)^3}$$

is nicely convergent *and* admits the easy change-of-variables argument for its periodicity.

[4.1] Claim: The function $\wp_\Lambda(z)$ is doubly-periodic, with period lattice Λ.

Proof: For $0 \neq \mu \in \Lambda$ the difference $\wp(z+\mu) - \wp(z)$ has derivative $\wp'(z+\mu) - \wp'(z) = 0$, by periodicity of $\wp'(z)$, so there is a constant C_μ such that $\wp(z+\mu) = \wp(z) + C_\mu$ for all z. Note that $\wp$ is an *even* function, because the term $1/z^2$ is invariant under $z \to -z$, and the other summands occur in pairs $(z\pm\lambda)^2 - \lambda^2$, interchanged by $z \to -z$. Take $z = -\mu/2$ to see that

$$C_\mu \;=\; \wp(-\mu/2+\mu) - \wp(\mu/2) \;=\; 0$$

proving the periodicity of $\wp$. ///

[4.2] Claim: An *entire* doubly-periodic function is *constant*.

Proof: Let ω_1, ω_2 be $\mathbb{Z}$-generators for Λ. Since the ω_i are linearly independent over $\mathbb{R}$, every $z \in \mathbb{C}$ is an $\mathbb{R}$-linear combination of them. Given $z = a\omega_1 + b\omega_2$ with $a, b \in \mathbb{R}$, let m, n be integers such that $0 \le a-m < 1$ and $0 \le b-n < 1$. Then

$$z \;=\; a\omega_1 + b\omega_2 \;=\; (a-m)\omega_1 + (b-n)\omega_2 + (m\omega_1 + n\omega_2)$$

Since $m\omega_1 + n\omega_2$ is in the lattice Λ, this shows that every Λ-orbit on $\mathbb{C}$ has a unique representative inside the so-called *fundamental domain*

$$F \;=\; \{r\omega_1 + s\omega_2 : 0 \le r < 1,\; 0 \le s < 1\}$$

for Λ. A Λ-periodic function's values on the whole plane are determined completely by its values on F. The set F has *compact* closure

$$\overline{F} \;=\; \{r\omega_1 + s\omega_2 : 0 \le r \le 1,\; 0 \le s \le 1\}$$

Thus, a continuous Λ-periodic function is *bounded* on $\overline{F}$, so bounded on $\mathbb{C}$. Thus, an entire Λ-periodic function is bounded. By Liouville's theorem, it is constant. ///

*[... iou ...]*pictures

[7] We follow Weierstraß's work on elliptic functions that came somewhat after Abel's and Jacobi's.

[4.3] Claim: Fix a lattice Λ. The Weierstraß P-function $\wp(z)$ and its derivative $\wp'(z)$ (attached to lattice Λ) satisfy the algebraic relation

$$\wp'(z)^2 \;=\; 4\wp(z)^3 - g_2\,\wp(z) - g_3$$

where

$$g_2 \;=\; g_2(\Lambda) = 60 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^4} \qquad g_3 \;=\; g_3(\Lambda) = 140 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^6}$$

[4.4] Remark: We will *find* the relation satisfied by $\wp$ and $\wp'$, not merely *verify* Weierstraß' relation, much as we did for singly-periodic functions.

Proof: The poles of both $\wp$ and $\wp'$ are just on the lattice Λ, so if we can make a linear combination of powers of $\wp$ and $\wp'$ whose Laurent expansion at 0 has no negative terms or constant term, then that linear combination of powers is identically 0.

Since $\wp(z)$ is *even*, the Laurent expansion of $\wp$ at 0 has no odd-degree terms. Because of the convergence trick, the constant Laurent coefficient of $\wp(z)$ at 0 is 0, so the expansion is of the form

$$\wp(z) \;=\; \frac{1}{z^2} + az^2 + bz^4 + O(z^6) \qquad \text{and} \qquad \wp'(z) \;=\; \frac{-2}{z^3} + 2az + 4bz^3 + O(z^5)$$

Then

$$\Big(\frac{\wp'(z)}{-2}\Big)^2 \;=\; \frac{1}{z^6} - \frac{2a}{z^2} - 4b + O(z) \qquad \text{and} \qquad \wp(z)^3 = \frac{1}{z^6} + \frac{3a}{z^2} + 3b + O(z)$$

so

$$\Big(\frac{\wp'}{-2}\Big)^2 - \wp^3 \;=\; \frac{-5a}{z^2} - 7b + O(z)$$

Then

$$\Big(\frac{\wp'}{-2}\Big)^2 - \wp^3 + 5a\,\wp + 7b = O(z)$$

As remarked at the beginning, this linear combination of powers is a doubly-periodic function without poles, so by Liouville is constant, yet vanishes at $z = 0$, so is 0. That is,

$$\wp'(z)^2 \;=\; 4\wp(z)^3 - 20a\,\wp(z) - 28b$$

With $\wp_o(z) = \wp(z) - \frac{1}{z^2}$,

$$a \;=\; \frac{\wp_o''(0)}{2!} \;=\; \frac{1}{2!}\cdot \sum_{0\neq\lambda\in\Lambda} \frac{(-2)(-3)}{\lambda^4} \;=\; 3 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^4}$$

and

$$b \;=\; \frac{\wp_o''''(0)}{2!} \;=\; \frac{1}{4!}\cdot \sum_{0\neq\lambda\in\Lambda} \frac{(-2)(-3)(-4)(-5)}{\lambda^6} \;=\; 5 \sum_{0\neq\lambda\in\Lambda} \frac{1}{\lambda^6}$$

we have Weierstraß'

$$\wp'(z)^2 \;=\; 4\wp(z)^3 - g_2\,\wp(z) - g_3$$

as anticipated. ///

[4.5] Remark: There is at least one other way to construct doubly-periodic functions directions, due to Jacobi, who expressed doubly-periodic functions as ratios of *entire* functions (*theta functions*) which are genuinely singly-periodic with periods (for example) $\mathbb{Z}$, and nearly (but not quite) periodic in another direction. (Indeed, we saw just above that entire functions that are genuinely doubly-periodic are constant!)

5. *Fields of elliptic functions*

[5.1] Theorem: *Every* elliptic function (with lattice Λ) is expressible in terms of the corresponding $\wp$ and $\wp'$. That is, for lattice Λ, the field of meromorphic Λ-periodic functions is exactly the collection of rational expressions in $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$. Further, all *even* Λ-periodic functions are rational expressions in $\wp_\Lambda(z)$.

Incidental to the proof of the theorem, we have

[5.2] Claim: Let f be a Λ-periodic meromorphic function. For a fixed choice of basis ω_1, ω_2 for Λ, let F be the corresponding *fundamental domain* as above. Let $z_1, \ldots, z_m$ be the zeros of f in F, and let $p_1, \ldots, p_n$ be the poles, both including multiplicities. [8] Then $m = n$. Further,

$$\sum_i z_i - \sum_j p_j \;=\; 0 \bmod \Lambda$$

Proof: Integrating f'/f around the boundary of F (make minor adaptations in case a zero or pole happens to be exactly on that path) computes $2\pi i(m-n)$, by Cauchy's residue theorem. On the other hand, by periodicity of f'/f, and since we integrate on opposite edges of the parallelogram F in opposite directions, this integral is 0. Thus, $m = n$.

Similarly, integrate $z \cdot f'/f$ around the boundary of F. On one hand, by Cauchy's residue theorem this computes

$$2\pi i \cdot \Big(\sum_i z_i - \sum_j p_j\Big)$$

This time, since the function with the factor of z thrown in is *not* periodic, the integral is not 0. However, there is still some cancellation. The integral is

$$-\omega_2 \int_0^{\omega_1} \frac{f'}{f} + \omega_1 \int_0^{\omega_2} \frac{f'}{f}$$

One may easily overlook the fact that the two integrals are *integer* multiples of $2\pi i$, which follows from [9]

$$\int_0^{\omega_i} \frac{f'}{f} \;=\; \int_0^{\omega_i} \frac{d\log f}{d\zeta}$$

and the fact that $f(0) = f(\omega_i)$. That is, as ζ goes from 0 to ω_i, the function $(\log f)(\zeta)$ traces out a closed path circling 0 some *integer* number of times, say k_i. Then the integral is

$$-\omega_2 \cdot 2\pi i k_1 + \omega_1 \cdot 2\pi i k_2 \in 2\pi i \cdot \Lambda$$

Cancelling the factor of $2\pi i$, equating the two outcomes gives

$$\sum_i z_i - \sum_j p_j \in \Lambda$$

as claimed. ///

[8] Usually *including multiplicities* means that for a zero z_o of order ℓ the point z_o is included ℓ times on the list of zeros. That is, this list is a *multiset*, not an ordinary set, since ordinary sets (by their nature) do not directly keep track of multiple occurrences of the same element.

[9] This is an instance of the *Argument Principle*.

Proof: Let f be a Λ-periodic meromorphic function on $\mathbb{C}$. We can break f into odd and even pieces by

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$$

For f *odd*, the function $\wp' \cdot f$ is *even*, so it suffices to prove that every *even* elliptic function is rational in $\wp$.

The previous claim has immediate implications for the values of $\wp$, which we use to form an expression in $\wp$ that will duplicate the zeros and poles of the given *even* f. Generally, for *even* f, since $f(-z) = f(z)$, for $2z_o \notin \Lambda$ and $f(z_o) = 0$, then $f(-z_o) = 0$ and z_o and $-z_o$ are distinct modulo Λ. For $2z_o \in \Lambda$, the *oddness* (and periodicity) of f' yields

$$f'(z_o) = -f'(-z_o) = -f'(-z_o + 2z_o) = -f'(z_o)$$

so $f'(z_o) = 0$, and the order of the zero z_o is at least 2.

In particular, by the previous claim, since $\wp(z) - \wp(a)$ has the obvious double pole on Λ, it has exactly two zeros, whose sum is 0 modulo Λ. Obviously a itself is a 0, and for $a \notin \frac{1}{2}\Lambda$ the unique (mod Λ) other zero is $-a$. And for $a \in \frac{1}{2}\Lambda$ it is a *double* zero of $\wp(z) - \wp(a)$.

Thus, for a zero $z_o \notin \Lambda$ of f, the order of vanishing of $\wp(z) - \wp(z_o)$ at *all* its zeros is *at most* that of f at those zeros. Thus, by comparison to $f(z)$, the function

$$\frac{f(z)}{\wp(z) - \wp(z_o)}$$

has lost two zeros (either z_o and $-z_o$ or a double zero at z_o). The double pole of $\wp(z) - \wp(z_o)$ at 0 makes $f(z)/(\wp(z) - \wp(z_o))$ have order of vanishing at 0 two more than that of $f(z)$. No new poles are introduced by such an alteration, nor any zeros off Λ. Thus, since there are only finitely-many zeros (modulo Λ), after finitely-many such modifications we have a function $g(z)$ with *no* zeros off Λ.

Next, we get rid of *poles* of $g(z)$ off Λ by a similar procedure, repeatedly *multiplying* by factors $\wp(z) - \wp(z_o)$. Thus, for some list of points z_i not in Λ, with positive and negative integer exponents e_i,

$$f(z) \cdot \prod_i \big(\wp(z) - \wp(z_i)\big)^{e_i}$$

has no poles or zeros off Λ. From the previous discussion, this expression has *no* zeros or poles at all, and then is constant. ///
