(*March 25, 2021*)

# 14a. Schwarz' lemma and automorphisms of $\mathfrak{H}$

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[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2020-21/14a_Schwarz_lemma.pdf]



Schwarz' Lemma is a fairly immediate consequence of the maximum modulus principle. Schwarz' lemma is the key technical point in classification of the holomorphic automorphisms of the upper half-plane $\mathfrak{H}$ (equivalently, of the open unit disk $\mathfrak{D}$). Namely, the linear fractional transformations

$$z \longrightarrow \frac{az+b}{cz+d} \qquad \left(\text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})\right)$$

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## 1. Schwarz' lemma

**[1.1] Theorem:** *(Schwarz)* Let $f$ be holomorphic on the open unit disk $\mathfrak{D}$, with $f(0) = 0$, and $|f(z)| \le 1$ for all $z \in \mathfrak{D}$. Then $|f(z)| \le |z|$ for all $z$, and $|f'(0)| \le 1$. Further, if either equality holds, then $f(z) = \alpha \cdot z$ with some $\alpha \in \mathbb{C}^\times$ with $|\alpha| = 1$.

*Proof:* Apply the maximum modulus principle to $g(z) = f(z)/z$, holomorphic on the disk because $f(0) = 0$. For fixed $z_o$ with $|z| < 1$, for any $r$ satisfying $|z_o| \le r < 1$, the maximum modulus principle gives

$$|g(z_o)| \;\le\; \sup_{|z|=r} |g(z)| \;\le\; \sup_{|z|=r} \frac{|f(z)|}{r} \;\le\; 1/r$$

Since this holds for every such $r$, in fact $|g(z)| \le 1$. That is, $|f(z)| \le |z|$ for $z \ne 0$. The limit at $z = 0$ gives $|f'(0)| \le 1$.

Further, if $|f(z_o)| = |z_o|$ for some $|z_o| < 1$, then $|g(z_o)| = 1$. Since $|g(z)| \le 1$ throughout $|z| < 1$, by the sharp form of the maximum modulus principle, $g(z)$ is a constant $\alpha$ with $|\alpha| = 1$. That is, $f(z)/z = \alpha$, and $f(z) = \alpha \cdot z$, with $|\alpha| = 1$.

Similarly, since $f'(0) = g(0)$, if $|f'(0)| = 1$, then $|g(0)| = 1$, and the same application of the sharp form of the maximum modulus principle shows that $f(z) = \alpha \cdot z$ with some $|\alpha| = 1$. ///

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## 2. Holomorphic automorphisms of $\mathfrak{D}$ fixing 0

**[2.1] Corollary:** Let $f$ be a holomorphic bijection of $\mathfrak{D}$ to itself, with $f(0) = 0$. Then $f^{-1}$ is also holomorphic, and $f(z) = \alpha \cdot z$ with some $|\alpha| = 1$.

*Proof:* For convenience, the lemma below recalls the argument that a bijective holomorphic function has a holomorphic inverse.

Schwarz' lemma gives $|f'(0)| \le 1$. Since $f$ is bijective, it is certainly bijective on a neighborhood of 0, so $f'(0) \ne 0$. By the holomorphic inverse function theorem, $f^{-1}$ is holomorphic on a neighborhood of 0, and

$(f^{-1})'(0) = 1/f'(0)$. Assuming $f^{-1}$ is holomorphic on the whole disk, $|(f^{-1})'(0)| \le 1$ by Schwarz' lemma. Thus, $|(f^{-1})'(0)| = |f'(0)| = 1$. Again by Schwarz' lemma, $f(z) = \alpha \cdot z$ with some $|\alpha| = 1$. ///

As invoked at the beginning of the proof:

**[2.2] Lemma:** A bijective holomorphic function $g$ (on a fixed open set) has a holomorphic inverse.

*Proof:* Holomorphy is a local property. Away from points $z_o$ where $g'(z_o) = 0$, the holomorphic inverse function theorem gives a local holomorphic inverse. It would suffice to know that $g$ cannot be bijective *locally* in a neighborhood of any point $z_o$ where $g'(z_o) = 0$. A stronger statement is of some interest:

**[2.3] Lemma:** Let $g$ be holomorphic on a neighborhood of $z_o$, and for some $n$

$$g(z_o) \;=\; g'(z_o) \;=\; g''(z_o) \;=\; \ldots \;=\; g^{(n-1)}(z_o) \qquad\qquad (n \ge 2)$$

Then $g$ is locally $n$-to-1 on a punctured neighborhood of $z_o$, in the following strong sense: There are sufficiently small $R > 0$ and $0 < r$ such that for $0 < |w_o| < r$ there are exactly $n$ points $z_1, \ldots, z_n$, in $0 < |z| \le R$ such that $g(z_j) = w_o$. The points $z_1, \ldots, z_n$ are *distinct*.

*Proof:* Without loss of generality $z_o = 0$. Multiplication by a (non-zero) constant also does not affect bijectivity. Thus, near $z_o = 0$,

$$g(z) \;=\; z^n + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} + \ldots \qquad\qquad (\text{with } |c_\ell| \le C \cdot M^\ell \text{ for some } C, M)$$

For $0 < R < \frac{1}{M}$, the circle $|z| = R$ is inside the disk of convergence of that power series. For $|z| \le R$,

$$\Big| g(z) - z^n \Big| \;\le\; |z|^n \sum_{\ell \ge n+1} C \cdot M^\ell \cdot R^{\ell - n} \;=\; |z|^n \cdot C \cdot \frac{M^{n+1} \cdot R}{1 - MR} \qquad\qquad (\text{in } |z| \le R)$$

Thus, for $R < \frac{1-MR}{C \cdot M^{n+1}}$, this is smaller than $|z|^n$. Thus, in $0 < |z| \le R$, $|g(z)| \ge |z|^n - |g(z) - z^n| > 0$. That is, $g(z)$ does not vanish in $0 < |z| \le R$. Visibly, $g$ vanishes to order $n$ at 0. (Rouché's theorem gives a similar conclusion.) Similarly, further shrinking $R$ if necessary, $g'(z)$ is also non-vanishing in $0 < |z| < R$.

Let $r = \min_{|z|=R} |g(z)|$. From the argument principle, for $|w_o| < r$,

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{g'(z)}{g(z) - w_o} \, dz \;=\; \text{number of times } g \text{ takes value } w_o \text{ inside } |z| = R$$

For $w_o = 0$, the value is $n$. In the region $|w_o| < r$, the integral is a continuous function of $w_o$ (in fact, holomorphic). But it can only take integer values. Since $\{w : |w| < r\}$ is connected, that integral must be constant on $|w_o| < r$. That is, each such value $w_o$ is taken exactly $n$ times in $|z| \le R$. Since the derivative $g'$ is non-vanishing on $0 < |z| \le R$, for $0 < |w_o| < r$ the points where $g$ takes value $w_o$ must be distinct.
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# 3. *Holomorphic automorphisms of $\mathfrak{H}$*

**[3.1] Corollary:** The linear fractional transformations $z \to \dfrac{az+b}{cz+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ give *all* the holomorphic automorphisms of $\mathfrak{H}$.

*Proof:*

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