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14b. Theta series, sums of squares

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The first (slightly frivolous) application here is to obtain a formula for the number of ways to express a positive integer as a sum of squares of eight integers. For example, we will prove

$$\text{number of ways to express odd prime } p \text{ as sum of 8 squares} \;=\; 16(p^3+1)$$

This will follow from the fact that the theta series

$$\theta(z) \;=\; \sum_{v\in\mathbb{Z}^8} e^{\pi i|v|^2 z} \qquad\qquad (\text{with } z\in\mathfrak{H})$$

is a holomorphic modular form of weight 4 for the subgroup

$$\Gamma_\theta \;=\; \Gamma(2)\cup w\Gamma(2) \qquad\qquad (\text{with } \Gamma(2)=\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod 2\})$$

where $w=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}$. Further, we use the fact that there are no weight 4 cuspforms for Γ_θ, from the *divisor formula* below. We also use the explicit computation of the Fourier coefficients of the two weight 4 Eisenstein series for Γ_θ.

One early modern study of representability by quadratic forms, by similar methods, was H.D. Kloosterman, *On the representation of numbers in the form* $ax^2+by^2+cz^2+dt^2$, Proc. London Math. Soc. **25** (1926), 143-173. Kloosterman recognized that *more* than four variables, especially an *even* number of variables, is much easier than four or fewer, and gave several references to earlier work.

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1. *Holomorphic modular forms for* Γ_θ

[1.1] Standard fundamental domain

The appendix proves that the translation $z\to z+2$ and the inversion $z\to -1/z$ generate Γ_θ, and there is the standard fundamental domain

$$F \;=\; \{z\in\mathfrak{H} \;:\; |z|\ge 1,\; -1\le z\le 1\}$$

There are two Γ_θ-inequivalent cusps at the boundary of the standard fundamental domain, $i\infty$ and 1, since -1 is identified with $+1$ by $z\to z+2$.

[1.2] No odd-weight modular forms for Γ_θ

Just as for $SL_2(\mathbb{Z})$, there are no (not-identically-zero) odd-weight holomorphic modular forms f for Γ_θ, since $-1_2\in\Gamma_\theta$, and

$$f(z) \;=\; f\Big|\begin{pmatrix} -1 & 0\\ 0 & -1\end{pmatrix}(z) \;=\; (-1)^k\cdot f(z)$$

[1.3] Fourier expansion at $i\infty$

Similar to $SL_2(\mathbb{Z})$, because of invariance under the translation $z \to z+2$, holomorphic modular forms for Γ_θ have Fourier series expansions of the form

$$f(z) \;=\; \sum_{n \geq \mathbb{Z}} c_n\, e^{\pi i n z}$$

with exponentials $e^{\pi i n z}$ rather than $e^{2\pi i n z}$ as for $SL_2(\mathbb{Z})$, since the latter contains the translation $z \to z+1$. That is, the *width* of the cusp $i\infty$ of Γ_θ is 2, not 1, as it was for $SL_2(\mathbb{Z})$.

Require that f is bounded as $y \to +\infty$. The Fourier coefficients are obtained by the expected formula, noting that the exponential involving y stays with the constant:

$$\tfrac{1}{2}\int_{-1}^{1} e^{-\pi i n x}\, f(x+iy)\, dx \;=\; \sum_m c_m \cdot \tfrac{1}{2}\int_{-1}^{1} e^{-\pi i n x}\, e^{\pi i m (x+iy)}\, dx$$

$$=\; \sum_m c_m \cdot \tfrac{1}{2}\int_{-1}^{1} e^{-\pi i n x}\, e^{\pi i m (x+iy)}\, dx \;=\; c_n \cdot e^{-2\pi n y}$$

Thus, the boundedness gives

$$|c_n\, e^{-2\pi n y}| \;=\; \Big|\tfrac{1}{2}\int_{-1}^{1} e^{-\pi i n x}\, f(x+iy)\, dx\Big| \;\leq\; \int_{-1}^{1} |f(x+iy)|\, dx \;\ll\; 1$$

For $n<0$, letting $y \to +\infty$ proves $c_n = 0$. The *order of vanishing* of f at $i\infty$ is the lowest index n_o such that $c_{n_o} \neq 0$. This much is parallel to $SL_2(\mathbb{Z})$.

[1.4] Boundedness at the other cusp 1 The details of the proof of the *divisor formula* below explain, at least with hindsight, that part of the definition of weight $2k$ *holomorphic modular form* f for Γ_θ require that, for $g \in SL_2(\mathbb{Z})$ with $g(i\infty)=1$, as $y \to +\infty$ the value $(f|_{2k}g)(z)$ is bounded.

[1.5] Remark: That is, as it happens, requiring that $f(z)$ itself be bounded as $z \to 1$ inside the standard fundamental region is *too strong* a condition. By accident, it would exclude some holomorphic Eisenstein series needed for a coherent discussion.

The action of $SL_2(\mathbb{C})$ on complex projective space $\mathbb{C} \cup \{\infty\}$ maps lines-and-circles to lines-and-circles, so any $g \in SL_2(\mathbb{R})$ mapping $i\infty$ to 1 maps regions $\{y > T\}$ to open disks tangent to the real line at 1. For fixed g, as T increases, the circles shrink. Thus, in this context, *approaching the cusp* 1 means to approach within such shrinking circles. In particular, this *excludes* tangential approach, or any approach other than asymptotically vertical.

[1.6] Fourier expansion at the other cusp 1 Also from the details of the *divisor formula* the *order of vanishing* of f at the cusp 1 should be expressed in terms of a *Fourier expansion* of f at 1. For this, we need to change coordinates on $\mathfrak{H}$ so that the isotropy group of 1 in Γ_θ becomes ordinary translations. That is, we want $g \in SL_2(\mathbb{Z})$ to map 1 to $i\infty$, since for $\gamma(1)=1$,

$$\gamma g^{-1}(i\infty) \;=\; \gamma(1) \;=\; 1 \;=\; g^{-1}(i\infty)$$

gives $(g\gamma g^{-1})(i\infty) = i\infty$. The isotropy group of $i\infty$ in $SL_2(\mathbb{R})$ is upper-triangular matrices P, so

$$g\big(\text{isotropy group of 1 in } \Gamma_\theta\big)g^{-1} \;\subset\; P$$

and, equivalently,

$$\text{isotropy group of 1 in } \Gamma_\theta \quad=\quad \Gamma_\theta \cap g^{-1} P g$$

An explicit choice of mapping 1 to $i\infty$ is easily accomplished by translating and inverting:

$$1 \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}(1) = 0 \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(0) = i\infty$$

That is, $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ maps $1 \to i\infty$. The *translations* in the isotropy group of 1 are $\Gamma_\theta \cap g^{-1}Ng$, where $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is the group of translations in $SL_2(\mathbb{R})$:

$$\text{translations in isotropy group of 1} \;=\; \Gamma_\theta \,\cap\, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \;=\; \begin{pmatrix} -b+1 & b \\ -b & b+1 \end{pmatrix}$$

The latter is in Γ_θ if and only if $b \in \mathbb{Z}$. That is, the *width* of the cusp 1 of Γ_θ is 1, in contrast to the cusp $i\infty$ which has width 2 for Γ_θ.

The Fourier expansion at 1 of a modular form f for Γ_θ is really the Fourier expansion of $(f|_{2k}g)(z)$ for $g \in SL_2(\mathbb{Z})$ with $g(i\infty) = 1$. The determination that the *width* is 1 shows that the expansion will be of the form

$$(f|_{2k}g)(z) \;=\; \sum_{n\in\mathbb{Z}} c_n\, e^{2\pi i n z}$$

The same argument as for $i\infty$ shows that boundedness implies that the negative-index Fourier coefficients are all 0.

[1.7] Hecke's estimate on Fourier coefficients of cuspforms Just as for $SL_2(\mathbb{Z})$, it is straightforward to obtain a useful asymptotic bound on Fourier coefficients of holomorphic cuspforms for Γ_θ. This will be used a little frivolously in obtaining an asymptotic for the number of ways to represent integers as sums of squares, and less friviolously in proving some equidistribution results on spheres.

[1.8] Theorem: *(Hecke)* For $f(z) = \sum_{n\ge1} c_n\, e^{\pi i n z}$ a holomorphic cuspform of weight $2k$ for Γ_θ,

$$c_m \;=\; O(n^{\frac{2k}{2}})$$

Proof: Since f is bounded as $y \to +\infty$,

$$|c_n \cdot e^{-\pi n y}| \;=\; \tfrac{1}{2}\Big|\int_0^2 e^{-\pi i n x}\, f(x+iy)\, dx\Big| \;\le\; \tfrac{1}{2}\int_0^2 |f(x+iy)|\, dx \; ll_f \; 1$$

This gives a bad preliminary estimate $|c_n| \ll_f e^{\pi n}$, by taking $y = 1$. As for $SL_2(\mathbb{Z})$, this gives exponential decay of $f(x+iy)$ as $y \to +\infty$:

$$|f(x+iy)| \;\le\; \sum_{n\ge1} |c_n|\, e^{-\pi n y} \; ll_f \; \sum_{n\ge1} e^{\pi n}\, e^{-\pi n y} \;=\; \frac{e^{-\pi(y-1)}}{1-e^{\pi(y-1)}}$$

A nearly identical discussion applies to f near the cusp 1, that is, with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$, the same discussion applies to $f|_{2k}g$ near $i\infty$. Namely,

$$\big(f|_{2k}g\big)(x+iy) \;\ll_f\; e^{-2\pi(y-1)} \qquad \text{(for } y \gg 1\text{)}$$

For example, with

$$g \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\big(f|_{2k}g\big)(x+iy) \;=\; z^{-2k}\cdot f(gz) \;\ll_f\; e^{-2\pi(y-1)} \;\ll_f\; e^{-2\pi(y-1)} \qquad\qquad (\text{for } y \gg 2)$$

Since the cusp 1 has width 1, it suffices to consider $0 \le x \le 1$, and in the region $0 \le x \le 1$ and $y \ge 2$,

$$|y| \;\ll\; |z|\; ll\; |y|$$

so

$$f(g(x+iy)) \;\ll\; y^{2k}\cdot|f(gz)| \;\ll_f\; y^{2k}\, e^{-2\pi(y-1)} \qquad\qquad (\text{in } 0\le x\le 1 \text{ and } y \ge 2)$$

The relevant qualitative point is that this is *of rapid decay* as $y \to \infty$. That is, $f(z)$ is of rapid decay as z approaches either cusp within a fixed fundamental domain.

Since f is continuous, $y^{\frac{2k}{2}}|f(z)|$ is *bounded* in the fundamental domain. It is also Γ_θ-invariant, so is bounded on the whole $\mathfrak{H}$.

Thus, as with $SL_2(\mathbb{Z})$, we boot-strap ourselves:

$$|c_n\cdot e^{-\pi n y}\cdot y^{\frac{2k}{k}}| \;=\; \tfrac{1}{2}y^{\frac{2k}{2}}\Big|\int_0^2 e^{-\pi i n x}\, f(x+iy)\,dx\Big| \;\le\; \tfrac{1}{2}\int_0^2 y^k\,|f(x+iy)|\,dx \;\ll_f\; 1$$

From $|c_n| \ll_f y^{-\frac{2k}{2}}e^{\pi n y}$ for all $y > 0$, optimize choice of y to minimize the upper bound: solve

$$0 \;=\; \frac{\partial}{\partial y}y^{-k}\,e^{\pi n y} \;=\; -ky^{-k-1}e^{\pi n y} + \pi n y^{-k}\,e^{\pi n y} \;=\; (-k+\pi n y)\cdot y^{-k-1}e^{\pi n y}$$

Thus, $y = k/\pi n$, giving

$$|c_n| \;\ll_f\; \Big(\frac{k}{\pi n}\Big)^{-\frac{2k}{2}} e^{\pi n\cdot k/\pi n} \;\ll_{f,k}\; n^{\frac{2k}{k}}$$

which is Hecke's estimate. ///

[1.9] Ramanujan's Delta In the discussion of $SL_2(\mathbb{Z})$, we saw that Ramanujan's cuspform $\Delta(z)$ of weight 12 does not vanish on $\mathfrak{H}$, but only at $i\infty$. It is certainly also a cuspform for Γ_θ. Since the *width* of the cusp 1 is just 1, $\Delta(z)$ vanishes to order 1 at the cusp 1. However, the width of the cusp $i\infty$ for Γ_θ is 2, so in the appropriate coordinates for Γ_θ, $\Delta(z)$ vanishes to order 2 at $i\infty$.

Apart from this possible surprise, there is the consequence that it is no longer possible to divide general cuspforms for Γ_θ by $\Delta(z)$ and obtain holomorphic modular forms, unless the given cuspform happens to vanish to order 2 at $i\infty$.

In discussion of the *divisor formula* below, we will discover that there *is* an essentially unique cuspform of weight 8, vanishing to first order at both cusps, and not vanishing at any point of $\mathfrak{H}$.

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2. *Holomorphic Eisenstein series for* Γ_θ

One use of holomorphic Eisenstein series is to subtract them from a given holomorphic modular form to produce a cuspform. That is, refering to the two (equivalence classes of) cusps by their representatives $i\infty$ and 1, for each weight $4 \le 2k \in 2\mathbb{Z}$, we want two Eisenstein series $E_{2k}^{(i\infty)}$ and E_{2k}^{1} so that, with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$,

$$\begin{cases} E_{2k}^{(i\infty)}(i\infty) &= 1 \\ E_{2k}^{(i\infty)}\big|_{2k}g(1) &= 0 \end{cases} \qquad\qquad \begin{cases} E_{2k}^{(1)}(i\infty) &= 0 \\ E_{2k}^{(1)}\big|_{2k}g(1) &= 1 \end{cases}$$

[2.1] Remark: That is, roughly, the Eisenstein series attached to a cusp σ should essentially have value 1 there and should essentially have value 0 at the other cusp. This does not quite literally happen, except at $i\infty$. Rather, the weight $2k$ action intervenes, reducing evaluation at other cusps to evaluation at $i\infty$.

[2.2] Moving Eisenstein series around For this subsection, we look at Eisenstein series defined by *congruence conditions* modulo N, because the patterns are clearer than if we'd just treat $N = 2$.

Fix $N \geq 1$, fix weight $4 \leq 2k \in 2\mathbb{Z}$, and for c_o, d_o mod N define a slightly different type of Eisenstein series

$$\widetilde{E}_{(c_o,d_o)}(z) \;=\; \sum_{(0,0)\neq(c,d),\;(c,d)=(c_o,d_o)\bmod N} \frac{1}{(cz+d)^{2k}} \qquad\qquad (\text{with } (c,d)\in\mathbb{Z}^2)$$

Generally, such an Eisenstein series will be a modular form only for the principal congruence subgroup

$$\Gamma(N) \;=\; \Gamma_N \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\}$$

One virtue of this description is the simplicity of behavior under the weight $2k$ action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the full group $SL_2(\mathbb{Z})$:

$$\widetilde{E}_{(c_o,d_o)}|_{2k}g(z) \;=\; (cz+d)^{-2k} \sum_{(0,0)\neq(m,n),\;(m,n)=(c_o,d_o)\bmod N} \frac{1}{(m\frac{az+b}{cz+d}+n)^{2k}} \;=$$

$$\sum_{(m,n)=(c_o,d_o)\bmod N} \frac{1}{(m(az+b)+n(cz+d))^{2k}} \;=\; \sum_{(m,n)=(c_o,d_o)\bmod N} \frac{1}{((ma+nc)z+(mb+nd))^{2k}}$$

Mod N we have

$$(\, ma+nc \quad mb+nd \,) \;=\; (\, m \quad n \,)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; (\, c_o \quad d_o \,)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod N$$

This gives the useful relation

$$\widetilde{E}_{(c_o,d_o)}|_{2k}g \;=\; \widetilde{E}_{(c_o,d_o)g}$$

In particular, the Fourier expansions of these Eisenstein series at various cusps are systematically accessible, as below. A more systematic version of the above description, for fixed level N and weight $2k$, is to let φ be a $\mathbb{C}$-valued function on $(\mathbb{Z}/N)^2$, and put

$$\widetilde{E}_\varphi(z) \;=\; \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \varphi(c,d)\,\frac{1}{(cz+d)^{2k}}$$

The same computation shows that, for $g \in SL_2(\mathbb{Z})$,

$$\widetilde{E}_\varphi|_{2k}g \;=\; \widetilde{E}_{\varphi\circ g^{-1}}$$

In effect, the previous version $\widetilde{E}_{c_o,d_o}$ used the function

$$\varphi(c,d) \;=\; \begin{cases} 1 & (\text{when } (c,d)=(c_o,d_o)\bmod N) \\ 0 & (\text{otherwise}) \end{cases}$$

The form $\widetilde{E}_\varphi$ is readily expressed in terms of the $\widetilde{E}_{c_o,d_o}$'s, by

$$\widetilde{E}_\varphi(z) \;=\; \sum_{(0,0)\neq(c,d)\in\mathbb{Z}^2} \varphi(c,d)\,\frac{1}{(cz+d)^{2k}} \;=\; \sum_{(c_o,d_o)\bmod N} \varphi(c_o,d_o) \sum_{0\neq(c,d)=(c_o,d_o)\bmod N} \frac{1}{(cz+d)^{2k}}$$

$$=\; \sum_{(c_o,d_o)\bmod N} \varphi(c_o,d_o)\cdot\widetilde{E}_{c_o,d_o}(z)$$

[2.3] Fourier expansions of Eisenstein series Description of Eisenstein series $\widetilde{E}$ as sums *with* congruence conditions but *without* the coprimality condition, as just above, facilitates determination of Fourier expansions.

Break the sum expressing the weight $2k$ level N Eisenstein series $\widetilde{E}_\varphi$ into subsums invariant under the translation $z \to z+N$ in $\Gamma(N)$:

$$\widetilde{E}_\varphi(z) \;=\; \sum_{0\neq(c,d)} \frac{\varphi(c,d)}{(cz+d)^{2k}} \;=\; \sum_{0\neq d} \frac{\varphi(0,d)}{d^{2k}} \;+\; \sum_{0\neq c}\sum_{d\in\mathbb{Z}} \frac{\varphi(c,d)}{(cz+d)^{2k}} \;=\; \sum_{0\neq d} \frac{\varphi(0,d)}{d^{2k}} + \sum_{0\neq c} \frac{1}{c^{2k}} \sum_{d\in\mathbb{Z}} \frac{\varphi(c,d)}{(z+\frac{d}{c})^{2k}}$$

The first of the latter two sums, being a constant, makes a contribution to the 0^{th} Fourier coefficient of $\widetilde{E}_{c_o,d_o}$. As with the level one holomorphic Eisenstein series, we will see that the rest of the sum does not contribute to the 0^{th} Fourier coefficient, that is, we anticipate that

$$\widetilde{E}_\varphi(i\infty) \;=\; \sum_{0\neq d} \frac{\varphi(0,d)}{d^{2k}}$$

For each $c \neq 0$, the inner sum over d can be rewritten by letting $d = d_1 + Nc\ell$ with $d_1 \in \mathbb{Z}/Nc$. Since $\varphi(c, Nc\ell + d_1)$ does not depend on ℓ,

$$\sum_{d\in\mathbb{Z}} \frac{\varphi(c,d)}{(z+\frac{d}{c})^{2k}} \;=\sum_{d_1\in\mathbb{Z}/Nc}\;\sum_{\ell\in\mathbb{Z}} \frac{\varphi(c,Nc\ell+d_1)}{(z+N\ell+\frac{d_1}{c})^{2k}} \;=\sum_{d_1\in\mathbb{Z}/Nc} \varphi(c,d_1) \sum_{\ell\in\mathbb{Z}} \frac{1}{(z+N\ell+\frac{d_1}{c})^{2k}}$$

The Fourier coefficients of the inner sum unwind:

$$\int_0^1 e^{-2\pi i n x/N} \sum_{\ell\in\mathbb{Z}} \frac{1}{(z+N\ell+\frac{d_1}{c})^{2k}}\, dx \;=\; \int_{\mathbb{R}} e^{-2\pi i n x/N} \frac{1}{(x+iy+\frac{d_1}{c})^{2k}}\, dx$$

As for Eisenstein series for $SL_2(\mathbb{Z})$, this can be evaluated by residues, treating x as a complex variable, replacing the integral along the real line by the limit of integrals $[-R,+R]$ and then over a large arc in either upper or lower half-plane, as $n \leq 0$ or $n > 0$, respectively. The pole at $x = -(iy+\frac{d_1}{c})$ is in the lower half-plane, so for $n \leq 0$ this is 0. For $n > 0$, because the curve is traced clockwise, it is

$$-2\pi i\,\mathrm{Res}_{x=-(iy+\frac{d_1}{c})} e^{-2\pi i n x/N} \frac{1}{(x+iy+\frac{d_1}{c})^{2k}} \;=\; \frac{1}{(2k-1)!}\Big(\frac{\partial}{\partial x}\Big)^{2k-1} e^{-2\pi i n x/N}\Big|_{x=-(iy+\frac{d_1}{c})}$$

$$=\; -2\pi i \frac{1}{(2k-1)!}(-2\pi i n/N)^{2k-1} \cdot e^{-2\pi i \frac{n}{N}\cdot(-(iy+\frac{d_1}{c}))} \;=\; \frac{(2\pi i)^{2k}}{(2k-1)!}(n/N)^{2k-1} \cdot e^{-2\pi \frac{n}{N} y} \cdot e^{2\pi i \frac{n}{N}\frac{d_1}{c}}$$

The exponential in y is as expected. In the sum

$$\sum_{d_1\in\mathbb{Z}/cN} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}}$$

the coefficient $\varphi(c,d_1)$ is unchanged under $d_1 \to d_1+N$, and the whole sum is stable under this change of variables, so [1]

$$\sum_{d_1\in\mathbb{Z}/cN} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}} \;=\; \sum_{d_1\in\mathbb{Z}/cN} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1+N}{c}} \;=\; e^{2\pi i\frac{n}{c}} \sum_{d_1\in\mathbb{Z}/cN} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}}$$

[1] This part of the computation is a reproof of an instance of the *cancellation lemma*.

Thus, the sum over d_1 is 0 unless $c|n$, and in the latter case

$$\sum_{d_1\in\mathbb{Z}/cN} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}} \;=\; |c|\cdot \sum_{d_1\in\mathbb{Z}/N} \varphi(c,d_1)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}} \qquad \text{(when } c|n\text{)}$$

Thus, for $n>0$, using the bijection $c\to n/c$ on divisors of n,

$$\int_0^1 e^{-2\pi i n x/N}\, \widetilde{E}_\varphi(x+iy)\, dx \;=\; \frac{(2\pi i)^{2k}}{(2k-1)!\, N^{2k-1}}\, e^{-2\pi n y/N} \sum_{0\neq c|n} \frac{n^{2k-1}}{c^{2k}}\, |c| \sum_{d_1\in\mathbb{Z}/N} \varphi\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{n}{N}\frac{d_1}{c}}$$

$$=\; \frac{(2\pi i)^{2k}}{(2k-1)!\, N^{2k-1}}\, e^{-2\pi n y/N} \sum_{0\neq c|n} |c|^{2k-1} \sum_{d_1\in\mathbb{Z}/N} \varphi\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{c}{N} d_1}$$

Since the total measure of $[0,N]$ is N, not 1, divide by one further power of N in the $n>0$ terms to correctly express the Fourier expansion in x, obtaining

$$\widetilde{E}_\varphi(x+iy) \;=\; \sum_{0\neq d} \frac{\varphi(0,d)}{d^{2k}} + \frac{(2\pi i)^{2k}}{(2k-1)!\, N^{2k}} \sum_{n>0} \Big(\sum_{0\neq c|n} |c|^{2k-1} \sum_{d_1\in\mathbb{Z}/N} \varphi\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{c}{N} d_1} \Big)\, e^{2\pi i n z/N}$$

[2.4] Back to Eisenstein series for Γ_θ

Recall the observation

$$\widetilde{E}_\varphi\Big|_{2k} g \;=\; \widetilde{E}_{\varphi\circ g^{-1}}$$

and the element

$$g \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;:\; i\infty \longrightarrow 1$$

From the previous computation, to make a Γ_θ Eisenstein series non-zero at $i\infty$ we want $\varphi(c,d)$ defined mod 2 such that

$$\varphi\circ w \;=\; \varphi \qquad \text{(so that } \widetilde{E}_\varphi|_{2k} w = \widetilde{E}_\varphi \text{ with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma_\theta\text{)}$$

and

$$\widetilde{E}_\varphi(i\infty) \;=\; \sum_d \frac{\varphi(0,d)}{d^{2k}} \;\neq\; 0$$

and

$$\widetilde{E}_{\varphi\circ g^{-1}}(i\infty) \;=\; \sum_{0\neq d} \frac{\varphi((0,d)g^{-1})}{d^{2k}} \;=\; 0$$

The space of functions on $(\mathbb{Z}/2)^2$ meeting the symmetry condition $\varphi\circ w = \varphi$ is three-dimensional space, spanned by

$$\varphi_1(c,d) \;=\; \begin{cases} 1 & \text{(for } (c,d) = (1,1) \bmod 2\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

$$\varphi_0(c,d) \;=\; \begin{cases} 1 & \text{(for } (c,d) = (0,0) \bmod 2\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

and

$$\varphi_2(c,d) \;=\; \begin{cases} 1 & \text{(for } (c,d) = (0,1) \text{ or } (1,0) \bmod 2\text{)} \\ 0 & \text{(otherwise)} \end{cases}$$

The data φ_0 gives an Eisenstein series $\widetilde{E}$ with non-zero value at $i\infty$, and is not changed by composition with g^{-1} since $(0,0)$ is stable, so gives a non-zero value at 1, as well.

The data φ_1 has the feature that $\varphi_1(0,d)=0$, so gives an Eisenstein series $\widetilde{E}$ with value 0 at $i\infty$. Composition with g^{-1} is understood by mapping $(1,1)$ under g, mod 2:

$$(1\quad 1)\cdot g \;=\; (1\quad 1)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (1\quad 0)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (0\quad 1) \qquad \pmod 2$$

Thus, the Eisenstein series $\widetilde{E}_{\varphi_1}$ has value at 1, in an appropriate sense, given by

$$(\text{suitable sense of value of } \widetilde{E}_{\varphi_1} \text{ at } 1) \;=\; \widetilde{E}_{\varphi_1}\Big|_{2k} g(i\infty) \;=\; \widetilde{E}_{0,1}(i\infty) \;=\; \sum_{d=1 \bmod 2} \frac{1}{d^{2k}} \;\neq\; 0$$

The value at $i\infty$ of $\widetilde{E}_{\varphi_2}$ is

$$\widetilde{E}_{\varphi_2}(i\infty) \;=\; \sum_{d=1 \bmod 2} \frac{1}{d^{2k}} \;\neq\; 0$$

To determine the suitable sense of the value of $\widetilde{E}_{\varphi_2}$ at 1, first observe

$$(0\quad 1)\cdot g \;=\; (0\quad 1)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (0\quad 1)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (1\quad 0) \qquad \pmod 2$$

and

$$(1\quad 0)\cdot g \;=\; (1\quad 0)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (1\quad 1)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \;=\; (1\quad 1) \qquad \pmod 2$$

Thus,

$$(\text{suitable sense of value of } \widetilde{E}_{\varphi_2} \text{ at } 1) \;=\; \widetilde{E}_{\varphi_2}\Big|_{2k} g(i\infty) \;=\; \widetilde{E}_{1,0}(i\infty) + \widetilde{E}_{1,1}(i\infty) \;=\; 0$$

In summary, up to normalizing constants, $\widetilde{E}_{\varphi_1}$ is the Eisenstein series *attached to* 1, and $\widetilde{E}_{\varphi_2}$ is the Eisenstein series *attached to* $i\infty$. In both the Fourier expansion of $E_{2k}^{(i\infty)}$ at $i\infty$ and in the Fourier expansion of $E_{2k}^{(1)}$ at 1, the leading non-zero constant is

$$\sum_{d=1 \bmod 2} \frac{1}{d^{2k}} \;=\; 2 \sum_{d\geq 1,\; d=1 \bmod 2} \frac{1}{d^{2k}} \;=\; 2\cdot\zeta(2k)\cdot(1-2^{-2k})$$

Dividing through by this constant to create values 1 at the corresponding cusps,

$$E_{2k}^{(i\infty)}(z) \;=\; 1 + \frac{(2\pi i)^{2k}}{(2k-1)!\,2^{2k}\,2\zeta(2k)(1-2^{-2k})} \sum_{n>0}\Big(\sum_{0\neq c|n} |c|^{2k-1} \sum_{d_1\in\mathbb{Z}/2} \varphi_2\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{c}{2} d_1}\Big)\, e^{2\pi i n z/2}$$

$$E_{2k}^{(1)}(z) \;=\; \frac{(2\pi i)^{2k}}{(2k-1)!\,2^{2k}\,2\zeta(2k)(1-2^{-2k})} \sum_{n>0}\Big(\sum_{0\neq c|n} |c|^{2k-1} \sum_{d_1\in\mathbb{Z}/2} \varphi_1\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{c}{2} d_1}\Big)\, e^{2\pi i n z/2}$$

These admit minor simplification. In both cases, changing the sign of c does not affect the inner sum, so the 2 in the denominators outside can be dropped while summing over $0<c|n$. In the case of φ_1, since $\varphi_1(\frac{n}{c},d)\neq 0$ only when both $\frac{n}{c}, d$ are *odd*, the inner sum over d_1 gives $(-1)^c$, so

$$E_{2k}^{(1)}(z) \;=\; \frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)(2^{2k}-1)} \sum_{n>0}\Big(\sum_{0<c|n,\; \frac{n}{c}\text{ odd}} c^{2k-1}\cdot(-1)^c\Big)\, e^{\pi i n z} \qquad (\text{for } \Gamma_\theta)$$

For non-vanishing of $\varphi_2(\frac{n}{c}, d_1)$, either $\frac{n}{c}$ is odd and d is even, or *vice-versa.* For $\frac{n}{c}$ odd and d_1 even, the exponential is 1 for all c. For $\frac{n}{c}$ even and d_1 odd, the exponential is $e^{\pi ic} = (-1)^c$. A formulaic interpolation of the situation is

$$\sum_{d_1 \in \mathbb{Z}/2} \varphi_2\Big(\frac{n}{c}, d_1\Big)\, e^{2\pi i \frac{c}{2} d_1} \;=\; (-1)^{(1+\frac{n}{c})\cdot c} \;=\; (-1)^{c+n} \;=\; (-1)^n \cdot (-1)^c$$

giving

$$E_{2k}^{(i\infty)}(z) \;=\; 1 + \frac{(2\pi i)^{2k}}{(2k-1)!\,\zeta(2k)(2^{2k}-1)} \sum_{n>0} (-1)^n \Big(\sum_{0<c|n} c^{2k-1} \cdot (-1)^c \Big)\, e^{\pi inz} \qquad (\text{for } \Gamma_\theta)$$

[2.5] Remark: The details of the *outcomes* of the previous computations are less important than the techniques for *doing* the computations. [2]

[2.6] Corollary: For a holomorphic modular form f of weight $2k \geq 4$ for Γ_θ, with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$,

$$f - f(i\infty) \cdot E_{2k}^{(i\infty)} - f|_{2k} g(i\infty) \cdot E_{2k}^{(1)}$$

is a cuspform of weight $2k$.

Proof: The restriction $2k \geq 4$ is for convergence of the Eisenstein series. The Eisenstein series are normalized to have (in a suitable sense) values 1 at their associated cusp and 0 (in a suitable sense) at the other cusp. In that sense, the evaluations of f and $f|_{2k}g$ at $i\infty$ are correct, as above. ///

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3. *Divisor formula for* Γ_θ

The divisor formula for weight $2k$ holomorphic modular forms f for $SL_2(\mathbb{Z})$ was

$$\frac{\nu_f(i)}{2} + \frac{\nu_f(\rho)}{3} + \nu_f(i\infty) + \sum_{\text{other } z} \nu_f(z) \;=\; \frac{2k}{12} \qquad (\text{for } SL_2(\mathbb{Z}))$$

with $\nu_f(z)$ the order of vanishing of f at z. This gave very strong constraints on spaces of level-one holomorphic modular forms of weight $2k$, in particular proving finite-dimensionality of each such space, an analogous result for Γ_θ is useful, although somewhat less decisively.

For a not-identically-zero holomorphic modular form f of weight $2k$ for Γ_θ, the order of vanishing $\nu_f(\subset \infty)$ is the smallest n_o such that $c_{n_o} \neq 0$ in the Fourier expansion

$$f(x+iy) \;=\; \sum_{n \geq 0} c_n \, e^{\pi inz}$$

and the order of vanishing $\nu_f(1)$ is the smallest n_o such that $b_{n_o} \neq 0$ in the Fourier expansion

$$f|_{2k}(x+iy) \;=\; \sum_{n \geq 0} b_n \, e^{2\pi inz}$$

The *divisor formula* is

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[2] For that matter, these techniques can be further improved by re-expressing Eisenstein series as functions on *adele groups* $GL_2(\mathbb{A})$, completely decoupling the archimedean and the various finite-prime parts of the computation.

[3.1] Theorem: For a not-identically-zero holomorphic modular form f of weight $2k$ for Γ_θ,

$$\frac{\nu_f(i)}{2} \;+\; \nu_f(i\infty) \;+\; \nu_f(1) \;+\; \sum_{\text{other } z} \nu_f(z) \quad=\quad \frac{2k}{4}$$

where the sum over *other* z is over mutually Γ_θ-inequivalent points in $\mathfrak{H}$ not Γ_θ equivalent to i.

Proof: As with $SL_2(\mathbb{Z})$, this follows from the *argument principle*, integrating the logarithmic deriative f'/f around the boundary σ of a truncated version of the fundamental domain $F = \{|z| \geq 1,\ |x| \leq 1\}$ for Γ_θ, and evaluating that integral in another way using the automorphy relations $f(z+2) = f(z)$ and $f(-1/z) = z^{2k} f(z)$. The argument principle gives *essentially*

$$\frac{1}{2\pi i}\int_\sigma \frac{f'(z)}{f(z)}\,dz \;=\; (\text{number of zeros inside } \sigma)$$

As for $SL_2(\mathbb{Z})$, there are some technicalities. First, points on the verticals $x = \pm 1$ can be accommodated by slighty indenting the path on $x = -1$ and correspondingly out-denting the path on $x = +1$, so that the modified path does not run right through any such 0. Similarly, any zeros on the arcs cutting off the cusps $i\infty$ and 1 can be dodged by moving the cut-off arc slightly. The point i requires more delicacy, since it is the fixed-point of $z \to -1/z$, so cannot be dodged so simply, and *half* the residue is picked up.

On the other hand, the automorphy conditions give much cancellation in the integral, as for $SL_2(\mathbb{Z})$. By $f(z+2) = f(z)$, certainly $f'(z+2) = f'(z)$, and since the $x = -1$ and $x = +1$ paths are traced in opposite directions, those parts of the whole integral *cancel* each other completely. The circular arcs from -1 to i and then i to $+1$ are mapped to each other by $z \to -1/z$, and directions are reversed, so we expect considerable cancellation. However, f and f' are not quite *invariant*: $f(-1/z) = z^{2k} \cdot f(z)$ gives

$$f'(-1/z)\frac{1}{z^2} \;=\; 2kz^{2k-1}f(z) + z^{2k}f'(z)$$

so

$$\frac{f'(-1/z)}{f(-1/z)}\,d(-1/z) \;=\; z^2 \cdot \frac{2kz^{2k-1}f(z) + z^{2k}f'(z)}{z^{2k}f(z)}\,\frac{dz}{z^2} \;=\; \Big(2kz^{-1} + \frac{f'(z)}{f(z)}\Big)\,dz$$

Thus, the sum of these two arc integrals almost cancels, leaving

$$-\frac{1}{2\pi i}\int_0^{\pi/2} 2k(e^{\pi i - it})^{-1}\;d(e^{\pi i - it}) \;=\; -\frac{1}{2\pi i}2k\int_0^{\pi/2}(e^{\pi i - it})^{-1}\;(-i)\,e^{\pi i - it}\;dt$$

$$=\; 2k\cdot\frac{\pi/2}{2\pi} \;=\; \frac{2k}{4}$$

The cusps are treated as for $SL_2(\mathbb{Z})$. At $i\infty$ for Γ_θ, the arc cutting off $i\infty$ at some height T captures the *negatives* of the orders of the zeros above that height, and the *negative* of $\nu_f(i\infty)$. Similarly at the cusp 1. Thus, refering only to z in the fundamental domain,

$$\nu_f(i) + \sum_{z \text{ inside path}} \nu_f(z) \;=\; \frac{2k}{4} - \nu_f(i\infty) - \nu_f(1) - \sum_{z \text{ outside path}} \nu_f(z)$$

which rearranges to the expected relation. ///

For Γ_θ and holomorphic modular forms:

[3.2] Corollary: The modular forms of weight 0 are constants. There are no cuspforms of weights 4 or 6. Up to constant multiples, there is a unique cuspform of weight 8, and this cuspform vanishes to first order at both cusps $i\infty$ and 1, and does not vanish on $\mathfrak{H}$. The weight 2 modular forms are multiples of $E_6^{(i\infty)}/E_4^{(i\infty)}$, and also multiples of $E_6^{(1)}/E_4^{(1)}$. For every weight $2k$ the space of holomorphic modular forms is finite-dimensional.

Proof: From the divisor formula, a weight 0 modular form f cannot vanish unless it vanishes identically, so $f - f(z_o)$ is identically 0, for any $z_o \in \mathfrak{H}$.

Since $2k/4 < 2$ for $2k = 4, 6$, and since a cuspform vanishes to order at least 1 at both cusps $i\infty$ and 1, there can be no cuspforms of weights 4 or 6.

Similarly, at weight $2k = 8$, we have $2k/4 = 2$, so the vanishing of a cuspform at the two cusps, would leave no room for any other vanishing. For existence, the *product* f of the two weight 4 Eisenstein series $E_4^{(i\infty)}$ and $E_4^{(1)}$ vanishes at both cusps, and is not identically 0. Thus, given a cuspform F of weight $2k \ge 8$, the quotient F/f is a modular form of weight $2k - 8$.

The general finite-dimensionality follows essentially by induction. Given a holomorphic modular form F of weight $2k \ge 8$, by subtracting multiples of the two Eisenstein series, without loss of generality F is a cuspform. Divide by the unique weight 8 cuspform so that F/f is of weight $2k - 8$. Thus, it suffices to prove finite-dimensionality for weights $0, 2, 4, 6$, already accomplished for weights $0, 4, 6$. The space of weight 2 modular forms F is mapped linearly to weight 6 modular forms by $F \to F \cdot E_4^{(i\infty)}$. Since $E_4^{(i\infty)}$ vanishes at the cusp 1, and F vanishes at i, these are the only zeros of this product. The Eisenstein series $E_6^{(i\infty)}$ has the same vanishing, so

$$\text{weight 2 modular form for } \Gamma_\theta \;=\; \text{(multiple of)}\; \frac{E_6^{(i\infty)}}{E_4^{(i\infty)}}$$

A similar argument applies with $E_6^{(1)}/E_4^{(1)}$. ///

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4. *Theta series and Poisson summation*

The basic *harmonic theta series* related to sums of n squares is

$$\theta_n(z) \;=\; \sum_{v\in\mathbb{Z}^n} e^{\pi i |v|^2 z} \;=\; \Big(\sum_{\ell\in\mathbb{Z}} e^{\pi i \ell^2 z}\Big)^n$$

For *odd* n this has more complicated behavior than for n *even*. The behavior is clearest for n divisible by 8, for reasons visible in the computation below, and we soon specialize to that case.

[4.1] Poisson summation, and inversion $z \to -1/z$ Fourier transform behaves well with respect to dilation: for $y > 0$, letting $(F \circ y)(x) = F(yx)$, a direct computation gives

$$(F\circ y)\widehat{\ }(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot x}\, F(yx)\, dx \;=\; \int_{\mathbb{R}^n} e^{-2\pi i \xi\cdot \frac{x}{y}}\, F(x)\, y^{-8n}\, dx$$

$$=\; y^{-n}\int_{\mathbb{R}^n} e^{-2\pi i \frac{\xi}{y}\cdot x}\, F(x)\, dx \;=\; y^{-n}\Big(\widehat{F}\circ \frac{1}{y}\Big)(\xi)$$

With y replaced by $\sqrt{y}$, invoking Poisson summation

$$\sum_{v\in\mathbb{Z}^n} \varphi(v) \;=\; \sum_{v\in\mathbb{Z}^n} \widehat{\varphi}(v)$$

compute

$$\theta_n(iy) \;=\; \sum_{v\in\mathbb{Z}^n} e^{-\pi|v|^2 y} \;=\; \frac{1}{\sqrt{y}^d}\sum_{v\in\mathbb{Z}^n} e^{-\pi|v\cdot\sqrt{y}|^2} \;=\; \frac{1}{\sqrt{y}^n}\cdot\sum_{v\in\mathbb{Z}^n} e^{-\pi|v/\sqrt{y}|^2}$$

$$=\; \frac{1}{(iy)^{\frac{n}{2}}}\sum_{v\in\mathbb{Z}^n} e^{-\pi|v|^2/y} \;=\; \frac{1}{(iy)^{\frac{n}{2}}}\cdot\theta_n(-1/iy)$$

[4.2] Remark: The complication of odd n is visible, namely, choosing a branch of square root. This is an *essential* complication, as it turns out that half-integral weight modular forms have very different behavior from integral-weight.

Thus, to avoid the half-integral weight complication, and to avoid a few further comparatively minor complications, replace n by $8n$. By the identity principle, from the corresponding identity for $y > 0$,

$$\theta_n(z) \;=\; \frac{1}{z^{4n}} \cdot \theta_n(-1/z)$$

This is the main issue in proving that $\theta_n(z)$ is a weight $4n$ modular form for Γ_θ. The lengths-squared $|v|^2$ are all integers, so the exponentials $e^{\pi i |v|^2 z}$ are all invariant under the translation $z \to z+2$. We showed that inversion and this translation generate Γ_θ.

[4.3] Behavior at cusp $i\infty$

Above, we showed that $\theta_f(z)$ is bounded as $y \to +\infty$. The defining expression for θ shows that its Fourier expansion at the cusp $i\infty$ has no negative-index coefficients.

[4.4] The other cusp 1

Checking the behavior of θ_n at the cusp 1 requires verifying that $\big(\theta_f|_{4n} g\big)(z)$ is bounded as $z \to +i\infty$, for $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$.

That is, the question is *not* simply about the values of $\theta_f(z)$ as z goes to 1 inside the fundamental domain, but about $\theta_f(z)$ altered by the weight $4n$ action.

With

$$g \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and noting the associativity

$$\theta_n|_{4n} g \;=\; \Big(\theta_n\Big|_{4n} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\Big)\Big|_{4n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

first compute

$$\Big(\theta_n\Big|_{4n} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\Big)(z) \;=\; \sum_v e^{\pi i |v|^2 (z+1)} \;=\; \sum_v e^{\pi i |v|^2}\, e^{\pi i |v|^2 z} \;=\; \sum_v (-1)^{|v|^2}\, e^{\pi i |v|^2 z}$$

That is, the summands are twisted by powers of -1, complicating application of Poisson summation to achieve the further effect of inversion $z \to -1/z$. Rewrite the sum as a sum over translates of lattices on each of which $e^{\pi i |v|^2} = (-1)^{|v|^2}$ is constant: replace v by $w + 2v$ with $v \in \mathbb{Z}^{8n}$ and $w \in \mathbb{Z}^{8n}$mod 2:

$$|w+2v|^2 \;=\; |w|^2 + 2\langle w, 2v\rangle + 4|v|^2 \;\in\; |w|^2 + 2\mathbb{Z}$$

so

$$\theta_n(z+1) \;=\; \sum_v e^{\pi i |v|^2}\, e^{\pi i |v|^2 z} \;=\; \sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} \sum_{v \in \mathbb{Z}^{8n}} e^{\pi i |2v+w|^2 z}$$

For each w, we will apply Poisson summation to the corresponding inner sum. Recall the behavior of Fourier transform on $\mathbb{R}^m$ under affine transformations $x \to ax + b$: by direct computation,

$$F(ax+b)\widehat{\ }(\xi) \;=\; \int_{\mathbb{R}^m} e^{-2\pi i \langle \xi, x\rangle}\, F(ax+b)\, dx \;=\; \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \xi, \frac{x}{a}\rangle}\, F(x+b)\, dx$$

$$= \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x\rangle}\, F(x+b)\, dx \;=\; \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x-b\rangle}\, F(x)\, dx$$

$$= \frac{1}{|a|^m}\, e^{2\pi i \langle \xi, \frac{b}{a}\rangle} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x\rangle}\, F(x)\, dx \;=\; \frac{1}{|a|^m}\, e^{2\pi i \langle \xi, \frac{b}{a}\rangle} \cdot \widehat{F}\big(\frac{\xi}{a}\big)$$

Applying this in Poisson summation, with $F(x) = e^{-\pi|x|^2}$ and $a = 2\sqrt{y}$ and $b = \sqrt{y}w$, the w^{th} inner sum at $z = iy$ is

$$\begin{aligned}\sum_{v\in\mathbb{Z}^{8n}} e^{-\pi|2v+w|^2 y} \;&=\; \sum_{v\in\mathbb{Z}^{8n}} e^{-\pi|2\sqrt{y}v+\sqrt{y}w|^2} \\ &=\; \frac{1}{(2\sqrt{y})^{8n}}\cdot \sum_{v\in\mathbb{Z}^{8n}} e^{2\pi i\langle v,\frac{\sqrt{y}w}{2\sqrt{y}}\rangle}\, e^{-\pi|v|^2/4y} \\ &=\; \frac{1}{(2\sqrt{y})^{8n}} \sum_{v\in\mathbb{Z}^{8n}} e^{2\pi i\langle v,\frac{w}{2}\rangle}\, e^{-\pi|v|^2/4y} \\ &=\; \frac{1}{(iy)^{4n}}\cdot\frac{1}{2^{8n}}\cdot \sum_{v\in\mathbb{Z}^{8n}} e^{\pi i\langle v,w\rangle}\, e^{-\pi|v|^2/4y}\end{aligned}$$

Summing over $w \in \mathbb{Z}^{8n}/2$ gives

$$\theta_n(iy+1) \;=\; \frac{1}{(iy)^{4n}\,2^{8n}} \sum_{v\in\mathbb{Z}^{8n}} \Big(\sum_{w\in\mathbb{Z}^{8n}/2} e^{\pi i|w|^2}\cdot e^{\pi i\langle v,w\rangle}\Big)\, e^{\pi i\frac{|v|^2}{4}(-1/iy)}$$

The identity principle gives the same identity for $z \in \mathfrak{H}$ in place of iy:

$$\theta_n(z+1) \;=\; \frac{1}{z^{4n}\,2^{8n}} \sum_{v\in\mathbb{Z}^{8n}} \Big(\sum_{w\in\mathbb{Z}^{8n}/2} e^{\pi i|w|^2}\cdot e^{\pi i\langle v,w\rangle}\Big) f(v)\, e^{\pi i\frac{|v|^2}{4}(-1/z)}$$

Replacing z by $-1/z$ and rearranging slightly,

$$z^{-4n}\cdot\theta_n\Big(\frac{-1}{z}+1\Big) \;=\; \frac{1}{2^{8n}} \sum_{v\in\mathbb{Z}^{8n}} \Big(\sum_{w\in\mathbb{Z}^{8n}/2} e^{\pi i|w|^2}\cdot e^{\pi i\langle v,w\rangle}\Big)\, e^{\pi i\frac{|v|^2}{4}z}$$

Without concern about further simplification, this exhibits the Fourier expansion of θ_n at 1, and there are no negative-index terms. In particular, as $y \to +\infty$, elementary estimates prove this is *bounded*, completing the proof that θ_n is a modular form of weight $4n$ for Γ_θ.

The 0^{th} Fourier coefficient of $\theta_n|_{4n}\,g$ is

$$\frac{1}{2^{8n}} \sum_{w\in\mathbb{Z}^{8n}/2} e^{\pi i|w|^2} \;=\; \frac{1}{2^{8n}}\Big(\sum_{w\in\mathbb{Z}/2} e^{\pi i w^2}\Big)^{8n} \;=\; \frac{1}{2^{8n}}\Big(1-1\Big)^{8n} \;=\; 0$$

That is, θ_n *vanishes* at the cusp 1.

[4.5] Remark: It is certainly not obvious from the original expression for $\theta_n(z)$ that the theta series vanishes (in a suitable sense) at cusp 1.

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5. *Sums of squares*

With $\nu_{8n}(N)$ the number of ways to express N as a sum of $8n$ squares of integers,

$$\theta_{8n}(z) \;=\; \sum_{v\in\mathbb{Z}^{8n}} e^{\pi i|v|^2 z} \;=\; \sum_{0\le N} \nu_{8n}(N)\, e^{\pi i N z}$$

This is a weight $4n$ holomorphic modular form for Γ_θ, and the previous section shows that it vanishes at the cusp 1. The value of $\theta_{8n}(z)$ at the cusp $i\infty$ is the 0^{th} Fourier coefficient, namely, 1.

[5.1] Eight squares The *divisor formula* showed that there are no cuspforms of weight 4 for Γ_θ. Thus, $\theta_8(z)$ is the Eisenstein series $E_4^{(i\infty)}(z)$ normalized so as to take value 1 at cusp $i\infty$ and 0 at cusp 1. Thus,

$$\nu_8(N) \;=\; N^{th} \text{ Fourier coefficient of } E_4^{(i\infty)} \;=\; \frac{(2\pi i)^4}{3!\,\zeta(4)(2^4-1)}\,(-1)^N \sum_{0<c|N} c^3\cdot(-1)^c$$

$$= \frac{16\pi^4}{6\cdot\frac{\pi^4}{90}\cdot 15}\,(-1)^N \sum_{0<c|N} c^3\cdot(-1)^c \;=\; 16\cdot(-1)^N \sum_{0<c|N} c^3\cdot(-1)^c$$

In fact, the value of the constant $\frac{(2\pi i)^4}{3!\,\zeta(4)(2^4-1)}$ could be *determined* by evaluating $\nu_8(1)$.

[5.2] $8n$ squares For $8n>8$, the weight $4n$ is above the range where there are no cuspforms for Γ_θ, so in general we invoke Hecke's estimate on Fourier coefficients of cuspforms, and can only can say

$$\nu_{8n}(N) \;=\; N^{th} \text{ Fourier coefficient of } \Big(E_{4n}^{(i\infty)} + \text{cuspform of weight } 4n\Big)$$

$$= \frac{(2\pi i)^{4n}}{(4n-1)!\,\zeta(4n)(2^{4n}-1)}\,(-1)^N\Big(\sum_{0<c|N} c^{4n-1}\cdot(-1)^c\Big) \quad+\quad O(N^{\frac{4n}{2n}})$$

[5.3] Corollary: For every $8n\ge 8$, there is a constant $C_n>0$, such that every sufficiently large $N\ge 1$ can be represented in at least $C_n\cdot N^{4n-1}$ ways as a sum of $8n$ squares.

Proof: The signed sum of powers of divisors function is *weakly multiplicative*, so it suffices to give a lower bound for prime powers p^e: first, for odd prime p,

$$(-1)^{p^e}\sum_{0<c|p^e} c^{4n-1}\,(-1)^c \;=\; (p^e)^{4n-1}-(p^{e-1})^{4n-1}+\ldots+(-1)^{p^e} \;\ge\; (p^e)^{4n-1}\cdot\Big(\frac{1}{1+p^{-(4n-1)}}\Big)$$

For $p=2$,

$$(-1)^{p^e}\sum_{0<c|p^e} c^{4n-1}\,(-1)^c \;=\; (p^e)^{4n-1}+(p^{e-1})^{4n-1}+\ldots+1 \;\ge\; \frac{(p^e)^{4n-1}}{1-p^{-(4n-1)}} \;\ge\; \frac{(p^e)^{4n-1}}{1+p^{-(4n-1)}}$$

The product of $1/(1+p^{-(4n-1)})$ over all primes is convergent to a finite non-zero constant B_n. Thus,

$$(-1)^N\Big(\sum_{0<c|N} c^{4n-1}\cdot(-1)^c\Big) \;\ge\; B_n\cdot N^{4n-1}$$

Further adjustment by the leading constant does not alter the qualitative conclusion. ///

[5.4] Corollary: A prime $p>2$ can be written as a sum of 8 squares in $16(p^3+1)$ ways. ///

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6. *Appendix: fundamental domain and generators for* Γ_θ

[6.1] Fundamental domain for Γ_θ and $\Gamma(2)$

The determination of the standard fundamental domain F for $\Gamma(1)=SL_2(\mathbb{Z})$ allows explicit determination of fundamental domains for finite-index *subgroups* such as the *principal congruence subgroups*

$$\Gamma(N) \;=\; \{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}) \;:\; \begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod N\}$$

by choosing coset representatives γ_i for $\Gamma(N)$ in $\Gamma(1)$, and then [3]

$$\text{fundamental domain for } \Gamma(N) \;=\; \bigcup_i \gamma_i F$$

It is useful that $\Gamma(N)$ is exactly the kernel of the group homomorphism

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N) \qquad \text{by} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a \bmod N & b \bmod N \\ c \bmod N & d \bmod N \end{pmatrix}$$

so is *normal* in $\Gamma(1)$.

Analytical methods in sums-of-squares problems use the important special choice

$$\Gamma_\theta \;=\; \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;:\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bmod 2\}$$

$$= \quad \Gamma(2) \;\; \cup \;\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \Gamma(2)$$

the coset-representative-oriented choice of fundamental domain can be adjusted to prove the corollary that Γ_θ is generated by $z \to -1/z$ and $z \to z+2$, as below.

[6.2] Remark: The following assertion holds without assuming p is prime, but all we need at the moment is $p = 2$, in any case. Further, the surjectivity of $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/2)$ is easy to observe directly, since, for example, the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

surject to $SL_2(\mathbb{Z}/2)$.

[6.3] Claim: For p prime, the natural map

$$SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/p) \qquad \text{is surjective}$$

Proof: Let q be the quotient map $\mathbb{Z} \to \mathbb{Z}/p$. First, given u, v not both 0 in $\mathbb{Z}/p$, we will find *relatively prime* c, d in $SL_2(\mathbb{Z})$ such that $qc = u$ and $qd = v$.

For $v \notin p\mathbb{Z}$, there is $0 \neq d \in R$ such that $qd = v$. Consider the conditions on $c \in R$

$$c = u \bmod p \qquad \text{and} \qquad c = 1 \bmod d$$

As $d \notin p\mathbb{Z}$, by the maximality of the ideal $p\mathbb{Z}$ there are $x \in \mathbb{Z}$ and $pm \in p\mathbb{Z}$ such that $xd + pm = 1$. Let $c = xdu + pm$. From $xd + pm = 1$, $xd = 1 \bmod pm$ and $pm = 1 \bmod d$, so this expression for c satisfies the two congruences conditions. In particular, $qc = u$, and since $c = 1 \bmod d$ it must be that $\gcd(c, d) = 1$.

For $v = 0$ in $\mathbb{Z}/p$, necessarily $u \neq 0$, and we reverse the roles of c, d in the previous paragraph.

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[3] Since $\mathfrak{H} = \bigcup_{\gamma \in \Gamma(1)} \gamma \overline{F}$, for representatives γ_i with $\Gamma(1) = \bigcup_i \Gamma(N)\gamma_i$,

$$\mathfrak{H} \;=\; \bigcup_{\gamma \in \Gamma(1)} \gamma \overline{F} \;=\; \bigcup_{\gamma \in \bigcup_i \Gamma(N)\gamma_i} \gamma \overline{F} \;=\; \bigcup_{\gamma \in \Gamma(N)} \bigcup_i \gamma\gamma_i \overline{F} \;=\; \bigcup_{\gamma \in \Gamma(N)} \gamma \Big(\bigcup_i \gamma_i \overline{F} \Big)$$

Thus, there are coprime c, d in $\mathbb{Z}$ whose images mod p are u, v. For integers s, t there exist a, b such that $\gcd(s,t) = as - bt$. The coprimality of c, d implies that there are a, b in R such that $ad - bc = 1$. That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ u & v \end{pmatrix} \bmod p$$

Further adjustment to accommodate the *upper* row is more straightforward: Given $\begin{pmatrix} r & s \\ u & v \end{pmatrix}$ in $SL_2(\mathbb{Z}/p)$, and letting $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ also denote its image in $SL_2(\mathbb{Z}/p)$,

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} v & -b \\ -u & a \end{pmatrix} = \begin{pmatrix} rv - su & * \\ uv - vu & * \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

The right-hand side is in $SL_2(\mathbb{Z}/p)$, so, in fact, it must be of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, and

$$\begin{pmatrix} r & s \\ u & v \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod p$$

So

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & s \\ u & v \end{pmatrix} \bmod p$$

giving the surjectivity. ///

[6.4] Claim: $\#SL_2(\mathbb{Z}/p) = (p^2 - 1)p$ for prime p.

Proof: First, count $GL_2(\mathbb{Z}/p)$. This is the number of ordered bases for the vector space $(\mathbb{Z}/p)^2$ over $\mathbb{Z}/p$, since an element of $GL_2(\mathbb{Z}/p)$ sends one basis to another, is transitive on ordered bases, and $g \in GL_2(\mathbb{Z}/p)$ fixes a basis v_1, v_2 only for $g = 1_2$.

The first basis element v_1 can be any non-zero vector in $(\mathbb{Z}/p)^2$, giving $p^2 - 1$ choices. For each such choice, the second basis element can be anything not on the $\mathbb{Z}/p$-line spanned by v_1, giving $p^2 - p$ choices. Thus, $\#GL_2(\mathbb{Z}/p) = (p^2 - 1)(p^2 - p)$.

The determinant map surjects $GL_2(\mathbb{Z}/p) \to (\mathbb{Z}/p)^\times$, and has kernel $SL_2(\mathbb{Z})$, so the *index* of $SL_2(\mathbb{Z}/p)$ is $\#(\mathbb{Z}/p)^\times = p - 1$, and the cardinality is as claimed. ///

[6.5] Corollary: $\Gamma(2)$ has six coset representatives in $\Gamma(1)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Proof: The index is $(2^2 - 1)2 = 6$. The six listed matrices are in $SL_2(\mathbb{Z})$ and are distinct mod 2. ///

[6.6] Corollary: Γ_θ has three coset representatives in $\Gamma(1)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Proof: The index is 3, since Γ_θ is index 2 above $\Gamma(2)$. The three listed matrices are in $SL_2(\mathbb{Z})$ and are not only distinct mod 2 but also do not differ mod $\Gamma(2)$ merely by multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. ///

[6.7] Corollary: A fundamental domain for Γ_θ is

$$F_\theta \;=\; \{z \in \mathfrak{H} \;:\; |z| \geq 1 \text{ and } |\mathrm{Re}(z)| \leq 1\}$$

Proof: With standard fundamental domain

$$F \;=\; \{z \in \mathfrak{H} \;:\; |z| \geq 1 \text{ and } |\mathrm{Re}(z) \leq \tfrac{1}{2}\}$$

for $\Gamma(1)$, the coset representatives for Γ_θ in $\Gamma(1)$ give a fundamental domain

$$F' \;=\; F \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F \cup \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} F$$

for Γ_θ. *[... iou ...]* pictures! We will symmetrize this into a more easily-describable form. With hindsight, we replace

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The point is that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F$ is understandable as a translate of the inverted F.

Move the *right half* of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} F \cup \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} F$ left by $z \to z - 2$, so that the two halves are symmetric about the imaginary axis. This produces the region claimed in the theorem. ///

[6.8] Generators for Γ_θ

[6.9] Corollary: Inversion $z \to -1/z$ and translation $z \to z + 2$ generate Γ_θ.

Proof: Given $z \in \mathfrak{H}$, translate z by $2\mathbb{Z}$ to $|\mathrm{Re}(z)| \leq 1$. If $|z| \geq 1$, stop. If not, invert, and then translate back to $|\mathrm{Re}(z)| \leq 1$. This produces a sequence of points $z_1, z_2, \ldots$ with

$$\mathrm{Im}(z_1) \;<\; \mathrm{Im}(z_2) \;<\; \ldots$$

As earlier, $\mathrm{Im}(z_n)$ is of the form $\mathrm{Im}(z)/|cz + d|^2$, and any such sequence must be finite. That is, inversion and translation by $1\mathbb{Z}$ eventually put z into the fundamental domain for Γ_θ.

Given $\gamma \in \Gamma_\theta$, choose z in the interior of the fundamental region, and let δ be a composition of inversions and translations by $2\mathbb{Z}$ so that $\delta^{-1}\gamma z$ is back in the fundamental domain. Then $\delta^{-1}\gamma = \pm 1_2$, so $\gamma = \pm\delta$. Since the inversion squares to -1_2, $\gamma \in \Gamma_\theta$. ///
