05b. Keyhole/Hankel contour and $\zeta(-n)$

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The contour-integration trick illustrated here appeared in one of Riemann’s proofs of analytic continuation of $\zeta(s)$. It almost immediately proves that values of $\zeta(s)$ at non-positive integers are rational, and shows the connection to the Laurent coefficients of $1/(e^t - 1)$ at $t = 0$.

[1.1] An integral representation of $\Gamma(s) \cdot \zeta(s)$ Although the integral representation of $\zeta(s)$ using a theta function is perhaps better in the long run, there is a more elementary one. As always,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{(in Re}(s) > 1)$$

[1.2] Claim: For Re$(s) > 1$,

$$\Gamma(s) \cdot \zeta(s) = \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t}$$

Proof: Expand a geometric series, exchange sum and integral, and change variables:

$$\int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t} = \int_0^\infty \frac{t^s e^{-t}}{1 - e^{-t}} \frac{dt}{t} = \int_0^\infty t^s \left(\sum_{n \geq 1} e^{-nt}\right) \frac{dt}{t} = \sum_{n \geq 1} \int_0^\infty t^s e^{-nt} \frac{dt}{t}$$

$$= \sum_{n \geq 1} \frac{1}{n^s} \int_0^\infty t^s e^{-nt} \frac{dt}{t} = \Gamma(s) \cdot \sum_{n \geq 1} \frac{1}{n^s} = \Gamma(s) \cdot \zeta(s)$$

as claimed. ///

[1.3] Keyhole/Hankel contour The keyhole or Hankel contour is a path from $+\infty$ inbound along the real line to $\varepsilon > 0$, counterclockwise around a circle of radius $\varepsilon$ at 0, back to $\varepsilon$ on the real line, and outbound back to $+\infty$ along the real line.

The usual elementary application is to evaluation of integrals similar to $\int_0^\infty \frac{t^s dt}{t^2 + 1}$, with $0 < \text{Re}(s) < 1$. In such an example, analytically continuing counterclockwise around 0 has no impact on the denominator, but, significantly, the numerator changes by a factor $e^{2\pi is}$, since

$$t^s = (|t| \cdot e^{i\theta})^s = |t|^s \cdot e^{i\theta s} \quad \text{(and } \theta \text{ goes from 0 to } 2\pi)$$

We want the out-bound value of $t^s$ to be real-valued for real $s$, so the inbound version of $t^s$ must be actually be $t^s \cdot e^{2\pi is}$.

The absolute value of the integrand goes to 0 as $|t| \to 0$, so the integral over the small circle goes to 0 as $\varepsilon \to 0$, as do the integrals to and from 0, $\varepsilon$ along the real line.

Thus, letting $H_\varepsilon$ be the Hankel contour with circle of radius $\varepsilon > 0$,

$$\lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^s dt}{t^2 + 1} = \lim_{\varepsilon \to 0} \left( \int_\varepsilon^\infty \frac{(t \cdot e^{2\pi i})^s dt}{t^2 + 1} + \text{(integral over little circle)} + \int_{\varepsilon}^{+\infty} \frac{t^s dt}{t^2 + 1} \right)$$

$$= (1 - e^{2\pi is}) \int_0^\infty \frac{t^s dt}{t^2 + 1}$$
The standard way to make the Residue Theorem useful is to modify \( H_z \) by not going all the way to \( +\infty \) outbound, but stopping at \(+R\) for large positive \( R \), traversing clockwise a large circle of radius \( R \) back to the positive real axis, and then inbound to \( \varepsilon \). We anticipate that the integrals from \( R \) to and from \( +\infty \) go to 0 as \( R \to +\infty \), as does the integral over the large circle.

A touch of care is necessary to correctly estimate \( z^s/(z^2 + 1) \) on \(|z| = R\), with our choice that

\[
(Re^{i\theta})^s = e^{s \log R \cdot e^{i\theta}} \quad \text{(with } 0 \leq \theta < 2\pi, \ R > 0, \ \log R \in \mathbb{R})
\]

In particular, the reliable conventional fact that \(|R^t| = 1\) for positive real \( R \) and real \( t \), is inadequate for a treatment of exponentials of complex numbers. For our specification of what \( z^s \) means in the present context, with \( z = Re^{i\theta}, \ R > 0, \ 0 \leq \theta < 2\pi, \) and \( s = u + iv \),

\[
|z^s| = |(R \cdot e^{i\theta})^s| = |R^s| \cdot |e^{i\theta s}| = R^{Re(s)} \cdot |e^{i\theta(u+iv)}|
\]

Since \( 0 \leq \theta < 2\pi \), \(|e^{-\theta v}| \leq e^{2\pi|v|}\). Since \( s = u + iv \) is fixed in this discussion, we have a uniform bound \( C = e^{2\pi|v|}\). Thus, on the circle \(|z| = R\),

\[
|z^s| = |(R \cdot e^{i\theta})^s| \leq R^{Re(s)} \cdot e^{2\pi|\Im(s)|}
\]

Specifically, for fixed \( s \) with \(-1 < \Re(s) < 1\), this is bounded by \( C \cdot R^1 \). Thus,

\[
\text{|integral over big circle|} \leq \text{length} \cdot \max \text{value} \leq 2\pi R \cdot C \cdot R^{Re(s)} = \frac{2\pi R}{R^2 - 1}
\]

For each \( R, \varepsilon \), this gives a path integral (counter-clockwise) over a closed path. By residues, this picks up \( 2\pi i \) times the sum of the residues inside the path. Thus, we discover that the integrals do not depend on the parameters \( 0 < \varepsilon < 1 < R \). Keeping track of the relevant versions of \( t^s \),

\[
(1 - e^{2\pi is}) \int_0^\infty \frac{t^s}{t^2 + 1} dt = 2\pi i \cdot \left( \text{(residue at } t = i) + \text{(residue at } t = -i) \right)
\]

\[
= 2\pi i \cdot \left( + \frac{e^{\frac{1}{2}\pi is}}{-i - i} + \frac{e^{\frac{3}{2}\pi is}}{i + i} \right) = \pi \cdot (e^{\frac{1}{2}\pi is} - e^{\frac{3}{2}\pi is})
\]

That is,

\[
\int_0^\infty \frac{t^s}{t^2 + 1} dt = \pi \cdot \frac{e^{\frac{1}{2}\pi is} - e^{\frac{3}{2}\pi is}}{1 - e^{2\pi is}} = \pi \cdot \frac{e^{-\frac{1}{2}\pi is} - e^{\frac{1}{2}\pi is}}{e^{-\pi is} - e^{\pi is}} = \frac{\pi}{e^{\frac{1}{2}\pi is} + e^{-\frac{1}{2}\pi is}} = \frac{2\pi}{\cos \frac{\pi}{2}}
\]

This is a charming and useful device, but a different secondary trick is applied to \( \zeta(s) \):

\[ \textbf{[1.4] Evaluation of } \zeta(-n) \quad \text{The first part of the Hankel contour discussion gives} \]

\[
\Gamma(s) \cdot \zeta(s) = \int_0^\infty \frac{t^s}{e^t - 1} \frac{dt}{t} = \frac{1}{1 - e^{2\pi is}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t} = \frac{1}{1 - e^{2\pi is}} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t}
\]

Rewrite this as

\[
\zeta(s) = \frac{1}{\Gamma(s) \cdot (1 - e^{2\pi is})} \cdot \lim_{\varepsilon \to 0} \int_{H_\varepsilon} \frac{t^s}{e^t - 1} \frac{dt}{t}
\]

At \( s = -n \in \{0, -1, -2, -3, -4, \ldots\} \) two fortunate things happen. First, the pole of \( \Gamma(s) \) and the zero of \( 1 - e^{2\pi is} \) cancel, giving a finite, computable value. Second, the function \( t^{-n-1} \) is single-valued, so the inbound and outbound integrals of the Hankel contour simply cancel each other, and the integral over the small circle at 0 becomes \( 2\pi i \) times the residue of \( \frac{t^{-n-1}}{e^t - 1} \) at 0.
The periodicity of $1 - e^{2\pi is}$ assures that the leading (linear) term in the power series at any integer is the same as that at 0, namely,

$$1 - e^{2\pi is} = 1 - \left(1 + \frac{2\pi is}{1!} + \frac{(2\pi is)^2}{2!} + \ldots\right) = 2\pi is + \text{higher}$$

Grant for the moment that the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$. Then

$$\zeta(-n) = \frac{1}{\Gamma(-n)} \cdot 2\pi i \cdot \text{Res}_{t=0} t^{n-1} e^t - 1 = (-1)^n \cdot n! \cdot \text{Res}_{t=0} t^{n-1} e^t - 1$$

The Laurent coefficients of $t^{n-1} e^t - 1$ are more-or-less Bernoulli numbers. These are not completely elementary objects, but are certainly rational. Thus, $\zeta(-n) \in \mathbb{Q}$.

**[1.5] Vanishing**

$\zeta(-2) = \zeta(-4) = \ldots = 0$  

A slightly finer analysis of the generating function $\frac{1}{1 - e^t}$ yields the vanishing of $\zeta(s)$ at negative even integers, as follows.

First, $\frac{1}{1 - e^t}$ is very close to being odd as a function of $t$:

$$\frac{1}{e^t - 1} + \frac{1}{e^{-t} - 1} = \frac{1}{e^t - 1} + \frac{e^t}{1 - e^t} = \frac{1}{e^t - 1} - \frac{e^t}{e^t - 1} = \frac{1 - e^t}{e^t - 1} = -1$$

Thus,

$$\left(\frac{1}{e^t - 1} + \frac{1}{2}\right) + \left(\frac{1}{e^{-t} - 1} + \frac{1}{2}\right) = 0$$

and $\frac{1}{e^t - 1} + \frac{1}{2}$ is odd, so all its non-vanishing Laurent coefficients are odd-degree. Thus, for even $-2n < 0$,

$$\zeta(-2n) = (-1)^{2n}(2n)! \cdot \text{Res}_{t=0} t^{2n-1} e^t - 1 = (2n)! \cdot (2n)^{th} \text{ Laurent coefficient of } \frac{1}{e^t - 1} = 0$$

**[1.6] Residues of $\Gamma(s)$**  

Finally, we determine the residues of $\Gamma(s)$. Certainly

$$\Gamma(1) = \int_0^\infty t^1 e^{-t} \frac{dt}{t} = \int_0^\infty e^{-t} \, dt = 1$$

From the functional equation $s\Gamma(s) = \Gamma(s+1)$, near $s = 0$

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{1 + \text{higher}}{s} = \frac{1}{s} + \text{(holomorphic at } s = 0)$$

Thus, the residue at 0 is 1. Iterating the functional equation,

$$\Gamma(s) = \frac{\Gamma(s+1)}{s} = \frac{\Gamma(s+2)}{(s+1)s} = \frac{\Gamma(s+3)}{(s+2)(s+1)s} = \ldots = \frac{\Gamma(s+n+1)}{(s+n)(s+n-1)\ldots(s+2)(s+1)s}$$

Thus, the leading Laurent term at $s = -n$ is

$$\left.\frac{\Gamma(s+n+1)}{(s+n-1)\ldots(s+2)(s+1)s}\right|_{s=-n} = \left.\frac{1}{s+n}\cdot\frac{\Gamma(-n+1)}{(-n+n-1)\ldots(-n+2)(-n+1)(-n)}\right.$$

That is, the residue of $\Gamma(s)$ at $-n$ is $(-1)^n/n!$ as claimed.

**Bibliography**
