

(February 9, 2022)

12. Weierstrass and Hadamard products

Paul Garrett garrett@math.umn.edu <http://www.math.umn.edu/~garrett/>

[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2021-22/12_Hadamard_products.pdf]

1. Weierstrass products
2. Poisson-Jensen formula
3. Hadamard products

Apart from factorization of polynomials, after Euler's

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

there is Euler's product for $\Gamma(z)$, which he used as the *definition* of the Gamma function:

$$\int_0^{\infty} e^{-t} t^z \frac{dt}{t} = \Gamma(z) = \frac{1}{z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}}$$

where the Euler-Mascheroni constant γ is essentially defined by this relation. The integral (Euler's) converges for $\operatorname{Re}(z) > 0$, while the product (Weierstrass') converges for all complex z except non-positive integers. Because the exponential factors are *linear*, and can *cancel*,

$$\frac{1}{\Gamma(z) \cdot \Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \cdot \sin \pi z$$

Linear exponential factors are exploited in Riemann's *explicit formula* [Riemann 1859], derived from equality of the *Euler product* and *Hadamard product* [Hadamard 1893] for the zeta function $\zeta(s) = \sum_n \frac{1}{n^s}$ for $\operatorname{Re}(s) > 1$:

$$\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \zeta(s) = \frac{e^{a+bs}}{s-1} \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right) e^{-s/2n}$$

where the product expansion of $\Gamma(\frac{s}{2})$ is visible, corresponding to *trivial zeros* of $\zeta(s)$ at negative even integers, and ρ ranges over all other, *non-trivial* zeros, known to be in the *critical strip* $0 < \operatorname{Re}(s) < 1$.

The hard part of the proof (below) of Hadamard's theorem is essentially that of [Ahlfors 1953/1966], with various rearrangements. A somewhat different argument is in [Lang 1993]. Some standard folkloric proofs of supporting facts about harmonic functions are recalled.

1. Weierstrass products

Given a sequence of complex numbers z_j with no accumulation point in \mathbb{C} , we will construct an entire function with zeros exactly the z_j .

[1.1] Basic construction

Taylor-MacLaurin polynomials of $-\log(1-z)$ will play a role: let

$$p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \dots + \frac{z^n}{n}$$

We will exhibit a sequence of integers n_j giving an *absolutely convergent infinite product* vanishing precisely at the z_j , with vanishing at $z=0$ accommodated by a suitable leading factor z^m , of the form

$$z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_{n_j}(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^{n_j}}{n_j z_j^{n_j}}\right)$$

Absolute convergence of $\sum_j \log(1+a_j)$ implies absolute convergence of the infinite product $\prod_j(1+a_j)$. Thus, we show that

$$\sum_j \left| \log\left(1 - \frac{z}{z_j}\right) + p_{n_j}\left(\frac{z}{z_j}\right) \right| < \infty$$

Fix a large radius R , keep $|z| < R$, and ignore the finitely-many z_j with $|z_j| < 2R$, so in the following $|z/z_j| < \frac{1}{2}$. Using the power series expansion of \log ,

$$\left| \log\left(1 - \frac{z}{z_j}\right) - p_n\left(\frac{z}{z_j}\right) \right| \leq \frac{1}{n+1} \cdot \left|\frac{z}{z_j}\right|^{n+1} + \frac{1}{n+2} \cdot \left|\frac{z}{z_j}\right|^{n+2} + \dots \leq \frac{1}{n+1} \cdot \frac{|z/z_j|^{n+1}}{1 - |z/z_j|} \leq 2 \cdot \frac{|z/z_j|^{n+1}}{n+1}$$

Thus, we want a sequence of positive integers n_j such that

$$\sum_{|z_j| \geq 2R} \frac{|z/z_j|^{n_j+1}}{n_j+1} < \infty \quad (\text{with } |z| < R)$$

The choice of n_j 's must be compatible with enlarging R , and this is easily arranged. For example, $n_j = j-1$ succeeds:

$$\sum_j \left|\frac{z}{z_j}\right|^j = \sum_{|z_j| < 2R} \left|\frac{z}{z_j}\right|^j + \sum_{|z_j| \geq 2R} \left|\frac{z}{z_j}\right|^j \leq \sum_{|z_j| < 2R} \left|\frac{z}{z_j}\right|^j + \sum_j 2^{-j}$$

Since $\{z_j\}$ is discrete, the sum over $|z_j| < 2R$ is finite, giving convergence, and convergence of the infinite product with $n_j = j$:

$$\prod_j \left(1 - \frac{z}{z_j}\right) e^{p_j(z/z_j)} = \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^j}{jz_j^j}\right)$$

[1.2] Canonical products and genus

Given entire f with zeros $z_j \neq 0$ and a zero of order m at 0, ratios

$$\varphi(z) = \frac{f(z)}{z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)}}$$

with *convergent* infinite products are *entire*, and *do not vanish*. Non-vanishing entire φ has an entire *logarithm*:

$$g(z) = \log \varphi(z) = \int_0^z \frac{\varphi'(\zeta) d\zeta}{\varphi(\zeta)}$$

Thus, non-vanishing entire φ is expressible as

$$\varphi(z) = e^{g(z)} \quad (\text{with } g \text{ entire})$$

Thus, the most general entire function with prescribed zeros is of the form

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_{n_j}(z/z_j)} \quad (\text{with } g \text{ entire})$$

With fixed f , altering the n_j necessitates a corresponding alteration in g .

We are most interested in zeros $\{z_j\}$ allowing a *uniform* integer h giving convergence of the infinite product in an expression

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

When f admits such a product expression with a uniform h , a product expression with *minimal* uniform h is a *canonical product* for f .

When, further, the leading factor $e^{g(z)}$ for f has $g(z)$ a *polynomial*, the *genus* of f is the maximum of h and the *degree* of g .

From the proof of the Weierstraß product theorem, we have

[1.3] **Corollary:** If an entire function is of genus h , then $\sum_n \frac{1}{|z_n|^{h+1}} < +\infty$, where the z_n are the zeros (with multiplicities), not including 0. Conversely, if $\sum_n \frac{1}{|z_n|^{h+1}} < +\infty$, then

$$\prod_n \left(1 - \frac{z}{z_n}\right) \cdot e^{p_h(z/z_n)}$$

converges nicely and gives an entire function of genus h . ///

2. Poisson-Jensen formula

Jensen's formula and the Poisson-Jensen formula are essential in the difficult half of the Hadamard theorem (below) comparing *genus* of an entire function to its *order of growth*.

The logarithm $u(z) = \log |f(z)|$ of the absolute value $|f(z)|$ of a non-vanishing holomorphic function f on a neighborhood of the unit disk is *harmonic*, that is, is annihilated by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$: expand

$$\Delta \log |f(z)| = \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right) \left(\frac{1}{2} \log f(z) + \frac{1}{2} \log \bar{f}(z)\right)$$

Conveniently, the two-dimensional Laplacian is the product of the Cauchy-Riemann operator and its conjugate. Since $\log f$ is holomorphic and $\log \bar{f}$ is *anti-holomorphic*, both are annihilated by the product of the two linear operators, so $\log |f(z)|$ is harmonic.

Thus, $\log |f(z)|$ satisfies the *mean-value property* for harmonic functions

$$\log |f(0)| = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

Next, let f have zeros ρ_j in $|z| < 1$ but none on the unit circle. We manufacture a holomorphic function F from f but without zeros in $|z| < 1$, and with $|F| = |f|$ on $|z| = 1$, by the standard ruse

$$F(z) = f(z) \cdot \prod_j \frac{1 - \bar{\rho}_j z}{z - \rho_j}$$

Indeed, for $|z| = 1$, the numerator of each factor has the same absolute value as the denominator:

$$|z - \rho_j| = \left| \frac{1}{z} - \bar{\rho}_j \right| = \frac{1}{|z|} \cdot |1 - \bar{\rho}_j z| = |1 - \bar{\rho}_j z|$$

For simplicity, suppose no ρ_j is 0. Applying the mean-value identity to $\log |F(z)|$ gives

$$\log |f(0)| - \sum_j \log |\rho_j| = \log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta$$

and then the basic *Jensen's formula*

$$\log |f(0)| - \sum_j \log |\rho_j| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta \quad (\text{for } |\rho_j| < 1)$$

The Poisson-Jensen formula is obtained by replacing 0 by an arbitrary point z inside the unit disk, by replacing f by $f \circ \varphi_z$, where φ_z is a *linear fractional transformation* mapping $0 \rightarrow z$ and stabilizing^[1] the unit disk:

$$\varphi_z = \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} : w \longrightarrow \frac{w + z}{\bar{z}w + 1}$$

This replaces the zeros ρ_j by $\varphi_z^{-1}(\rho_j) = \frac{\rho_j - z}{-\bar{z}\rho_j + 1}$. Instead of the mean-value property expressing $f(0)$ as an integral over the circle, use the *Poisson formula* for $f(z)$. This gives the basic *Poisson-Jensen formula*

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{z}\rho_j + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < 1, |\rho_j| < 1)$$

Generally, for holomorphic f on a neighborhood of a disk of radius $r > 0$ with zeros ρ_j in that disk, apply the previous to $f(r \cdot z)$ with zeros ρ_j/r in the unit disk:

$$\log |f(r \cdot z)| - \sum_j \log \left| \frac{\rho_j/r - z}{-\bar{z}\rho_j/r + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r \cdot e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < 1)$$

Replacing z by z/r gives

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j/r - z/r}{-\bar{z}\rho_j/r^2 + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{1 - |z/r|^2}{|z/r - e^{i\theta}|^2} d\theta \quad (\text{for } |z| < r)$$

[1] To verify that such maps stabilize the unit disk, expand the natural expression:

$$\begin{aligned} 1 - \left| \frac{w + z}{\bar{z}w + 1} \right|^2 &= |\bar{z}w + 1|^{-2} \cdot \left(|\bar{z}w + 1|^2 - |w + z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot \left(|zw|^2 + \bar{z}w + z\bar{w} + 1 - |w|^2 - \bar{z}w - z\bar{w} - |z|^2 \right) \\ &= |\bar{z}w + 1|^{-2} \cdot \left(|zw|^2 + 1 - |w|^2 - |z|^2 \right) = |\bar{z}w + 1|^{-2} \cdot (1 - |z|^2) \cdot (1 - |w|^2) > 0 \end{aligned}$$

which rearranges to the general *Poisson-Jensen formula*

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\bar{z}\rho_j/r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta \quad (\text{for } |z| < r, |\rho_j| < r)$$

The case $z = 0$ is the general *Jensen formula* for arbitrary radius r :

$$\log |f(0)| - \sum_j \log \left| \frac{\rho_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad (\text{with } |\rho_j| < r)$$

3. Hadamard products

The *order* of an entire function f is the smallest positive real λ , if it exists, such that, for every $\varepsilon > 0$,

$$|f(z)| \leq e^{|z|^{\lambda+\varepsilon}} \quad (\text{for all sufficiently large } |z|)$$

Recall that, in an infinite product expression with compensating exponential factors with *uniform degree* h

$$f(z) = e^{g(z)} \cdot z^m \cdot \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

when the exponent $g(z)$ is *polynomial*, the *genus* of f is the maximum of h and the *degree* of g .

[3.1] Theorem: (*Hadamard*) The *genus* h and order λ are related by $h \leq \lambda < h + 1$. In particular, one is *finite* if and only the other is finite.

Proof: First, the easier half. For f of finite genus h expressed as

$$f(z) = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp\left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \dots + \frac{z^h}{hz_j^h}\right)$$

the leading exponent is polynomial g of degree at most h , so $e^{g(z)}$ is of *order* at most h . The order of a product is at most the maximum of the orders of the factors, so it suffices to prove that the order of the infinite product is at most $h + 1$.

The assumption that h is the genus of f is equivalent to

$$\sum_j \frac{1}{|z_j|^{h+1}} < \infty$$

We use this to directly estimate the infinite product and show that it has order of growth $\lambda < h + 1$.

Toward an estimate on $F_h(w) = (1 - w)e^{p_h(w)}$ applicable for *all* w , not merely for $|w| < 1$, we collect some inequalities. There is the basic

$$\log |F_h(w)| = \log |(1 - w)e^{p_{h-1}(w)} \cdot e^{w^h/h}| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \quad (\text{for all } w)$$

As before, for $|w| < 1$,

$$\log |F_h(w)| \leq \frac{1}{h+1} \cdot |w|^{h+1} + \frac{1}{h+2} \cdot |w|^{h+2} + \dots \leq |w|^{h+1} \cdot \frac{1}{1 - |w|} \quad (\text{for } |w| < 1)$$

This gives $(1 - |w|) \cdot \log |F_h(w)| \leq |w|^{h+1}$ for $|w| < 1$. Adding to the latter the basic relation multiplied by $|w|$ gives

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } |w| < 1)$$

In fact, the latter inequality also holds for $|w| \geq 1$ and $\log |F_{h-1}(w)| \geq 0$, from the basic relation. For $\log |F_{h-1}(w)| < 0$ and $|w| \geq 1$, from the basic relation,

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \leq \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } \log |F_{h-1}(w)| < 0 \text{ and } |w| \geq 1)$$

Now prove $\log |F_h(w)| \ll_h |w|^{h+1}$, by induction on h . For $h = 0$, from $\log |x| \leq |x| - 1$,

$$\log |1 - w| \leq |1 - w| - 1 \leq 1 + |w| - 1 = |w|$$

Assume $\log |F_{h-1}(w)| \ll_h |w|^h$. For $|w| < 1$, we reach the desired conclusion by

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + \left(1 + \frac{1}{h}\right) |w|^{h+1} \ll_h |w| \cdot |w|^h + \left(1 + \frac{1}{h}\right) |w|^{h+1} \quad (\text{for } |w| < 1)$$

For $|w| \geq 1$ and $\log |F_{h-1}(w)| > 0$, from the basic relation

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h + \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| > 0)$$

For $\log |F_{h-1}(w)| \leq 0$ and $|w| \geq 1$, from the basic relation we already have

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad (\text{for } |w| \geq 1 \text{ and } \log |F_{h-1}(w)| < 0)$$

This proves $\log |F_h(w)| \ll_h |w|^{h+1}$ for all w .

Estimate the infinite product:

$$\log \left| \prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{p_h(z/z_j)} \right| = \sum_j \log \left| \left(1 - \frac{z}{z_j}\right) \cdot e^{p_h(z/z_j)} \right| \ll_h \sum_j \left| \frac{z}{z_j} \right|^{h+1} < \infty$$

since $\sum 1/|z_j|^{h+1}$ converges. Thus, such an infinite product has *growth order* $\lambda \leq h + 1$.

Now the difficult half of the proof. Let $h \leq \lambda < h + 1$. *Jensen's formula* will show that the zeros z_j are sufficiently spread out for convergence of $\sum 1/|z_j|^{h+1}$. Without loss of generality, suppose $f(0) \neq 0$. From

$$\log |f(0)| - \sum_j \log \left| \frac{z_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad (\text{with } |z_j| < r)$$

certainly

$$\sum_{|z_j| < r/2} \log 2 \leq \sum_{|\rho_j| < r/2} -\log \left| \frac{\rho_j}{r} \right| \ll_\varepsilon -\log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} r^{\lambda+\varepsilon} d\theta \ll r^{\lambda+\varepsilon} \quad (\text{for every } \varepsilon > 0)$$

With $\nu(r)$ the number of zeros inside the disk of radius r , this gives

$$\lim_{r \rightarrow +\infty} \frac{\nu(r)}{r^{\lambda+\varepsilon}} = 0 \quad (\text{for all } \varepsilon > 0)$$

Order the zeros by absolute value: $|z_1| \leq |z_2| \leq \dots$ and for simplicity suppose no two have the same size. Then $j = \nu(|z_j|) \ll_\varepsilon |z|^{h+\varepsilon}$. Thus,

$$\sum \frac{1}{|z_j|^{h+1}} \ll_\varepsilon \sum \frac{1}{(j^{\frac{1}{\lambda+\varepsilon}})^{h+1}} = \sum \frac{1}{j^{\frac{h+1}{\lambda+\varepsilon}}}$$

The latter converges for $\frac{h+1}{\lambda+\varepsilon} > 1$, that is, for $\lambda + \varepsilon < h + 1$. When $\lambda < h + 1$, there is $\varepsilon > 0$ making such an equality hold.

It remains to show that the entire function $g(z)$ in the leading exponential factor is of degree at most $h + 1$, by showing that its $(h + 1)^{th}$ derivative is 0.

In the Poisson-Jensen formula

$$\log |f(z)| - \sum_{|z_j| < r} \log \left| \frac{z_j - z}{-\bar{z}z_j/r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} d\theta \quad (\text{for } |z| < r)$$

application of $\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ annihilates the anti-holomorphic parts, returning us to an equality of holomorphic functions, as follows. The effect on the integrand is

$$\begin{aligned} 2 \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} &= 2 \frac{-\bar{z}}{(z - re^{i\theta})(\bar{z} - re^{-i\theta})} - \frac{r^2 - |z|^2}{(z - re^{i\theta})^2 (\bar{z} - re^{-i\theta})} \\ &= 2 \frac{-|z|^2 + \bar{z}re^{i\theta} - r^2 + |z|^2}{(z - re^{i\theta})^2 (\bar{z} - re^{-i\theta})} = 2 \frac{re^{i\theta}}{(re^{i\theta} - z)^2} \end{aligned}$$

Thus,

$$\frac{f'(z)}{f(z)} - \sum_{|z_j| < r} \frac{1}{z - z_j} + \sum_{|z_j| < r} \frac{\bar{z}_j}{\bar{z}_j z - r^2} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^2} d\theta$$

Further differentiation h times in z gives

$$\left(\frac{f'(z)}{f(z)} \right)^{(h)} = \sum_{|z_j| < r} \frac{(-1)^h h!}{(z - z_j)^{h+1}} - \sum_{|z_j| < r} \frac{(-1)^h h! \cdot \bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} + \frac{(-1)^h (h+1)!}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta$$

We claim that the second sum and the integral go to 0 as $r \rightarrow +\infty$.

Regarding the integral, Cauchy's integral formula gives

$$\int_0^{2\pi} \frac{re^{i\theta}}{(z - re^{i\theta})^{h+2}} d\theta = 0$$

Letting M_r be the maximum of $|f|$ on the circle of radius r , taking $|z| < r/2$, up to sign the integral is

$$\int_0^{2\pi} \log \left(\frac{M_r}{|f(re^{i\theta})|} \right) \cdot \frac{2re^{i\theta} d\theta}{(z - re^{i\theta})^{h+2}} \ll \frac{1}{r^{h+1}} \int_0^{2\pi} \log \left(\frac{M_r}{|f(re^{i\theta})|} \right) d\theta \ll_\varepsilon \frac{r^{\lambda+\varepsilon}}{r^{h+1}} \cdot \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta$$

Jensen's formula gives

$$\frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta \leq -\log |f(0)|$$

Thus, for $\lambda + \varepsilon < h + 1$ the integral goes to 0 as $r \rightarrow +\infty$.

For the second sum, again take $|z| < r/2$, so for $|z_j| < r$

$$\left| \frac{\bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} \right| \leq \frac{|z_j^{h+1}|}{(r^2 - |z_j| \cdot \frac{r}{2})^{h+1}} \ll \frac{|z_j^{h+1}|}{r^{h+1} (r - |z_j|)^{h+1}} \ll \frac{1}{r^{h+1}}$$

We already showed that the number $\nu(r)$ of $|z_j| < r$ satisfies $\lim \nu(r)/r^{h+1} = 0$. Thus, this sum goes to 0 as $r \rightarrow +\infty$. Taking the limit,

$$\left(\frac{f'}{f}\right)^{(h)} = (-1)^h h! \sum_j \frac{1}{(z - z_j)^{h+1}}$$

Returning to $f(z) = e^{g(z)} \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_h(z/z_j)}$, taking logarithmic derivative gives

$$\frac{f'}{f} = g' + \sum_j \left(\frac{1}{z - z_j} + \frac{p'_h(z/z_j)}{z_j} \right)$$

and taking h further derivatives gives

$$\left(\frac{f'}{f}\right)^{(h)} = g^{(h+1)} + \sum_j \frac{(-1)^h h!}{(z - z_j)^{h+1}}$$

Since the h^{th} derivative of f'/f is the latter sum, $g^{(h+1)} = 0$, so g is a polynomial of degree at most h .
 ///

[Ahlfors 1953/1966] L. Ahlfors, *Complex Analysis*, McGraw-Hill, 1953, second edition, 1966.

[Hadamard 1893] J. Hadamard, *Étude sur les Propriétés des Fonctions Entières et en Particulier d'une Fonction Considérée par Riemann*, J. Math. Pures Appl. **1893**, 171-215.

[Lang 1993] S. Lang, *Complex Analysis*, Springer-Verlag, 1993.

[Riemann 1859] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monats. Akad. Berlin (1859), 671-680.

[Whittaker-Watson 1927] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1927, 4th edition, 1952.