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05f. Counting zeros of $\zeta(s)$

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We count zeros of $\zeta(s)$ in the critical strip $0 \le \mathrm{Re}(s) \le 1$ [1] using the *argument principle* and the Laplace-Stirling asymptotic

$$\Gamma(s) \;=\; (s-\tfrac{1}{2})\log s - s + O(1) \qquad\qquad (\text{in } \mathrm{Re}(s) \ge \delta > 0 \text{ as } |s| \to \infty)$$

The convergent Euler product shows that there are no zeros in $\mathrm{Re}(s) > 1$. The functional equation $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$ shows that the only zeros of $\zeta(s)$ in $\mathrm{Re}(s) < 0$ are where $\Gamma(\frac{s}{2})$ has poles, namely, negative even integers. These are the *trivial* zeros of $\zeta(s)$. The counting function of interest is

$$N(T) \;=\; \text{number of zeros of } \zeta(s) \text{ in } 0 \le \mathrm{Im}(s) \le T \text{ and } 0 \le \mathrm{Re}(s) \le 1$$

For any fixed $1 < 2 \in \mathbb{R}$, with $2 \notin \mathbb{Z}$, by the *argument principle* [2]

$$N(T) \;=\; \tfrac{1}{2} \cdot \Big(\frac{1}{2\pi i}\int_{R_T} \frac{\xi'(s)}{\xi(s)}\, ds\Big) \;+\; O(1)$$

where R_T is the rectangle connecting $2 \pm iT$ and $1-2 \pm iT$, traversed counter-clockwise, deformed slightly to skirt any zeros of $\zeta(s)$. The division by 2 takes into account the double-counting of zeros off the real interval $[0,1]$, and the $O(1)$ term accounts for the pole at $s = 1$. [3] The following is an estimate of this integral, giving the leading terms in an asymptotic in T.

[0.1] Theorem: $N(T) \;=\; \dfrac{1}{2\pi} \cdot T\log T \;-\; \dfrac{\log 2\pi e}{2\pi} \cdot T \;+\; O(\log T) \;\;=\;\; \dfrac{1}{2\pi} T \log \dfrac{T}{2\pi e} + O(\log T)$

[0.2] Remark: The vertical asymptotics of $\Gamma(s)$ completely determine the leading terms of the asymptotic expansion, by a direct computation which determines the constants.

Proof: Using the functional equation $\xi(1-s) = \xi(s)$, and the symmetry $\xi(\overline{s}) = \overline{\xi(s)}$, we integrate only upward from 2 to $2+iT$, and then left from $2+iT$ to $\frac{1}{2}+iT$. The argument-principle integral computes $1/2\pi$ times the net change in the imaginary part of $\log\xi(s)$ over the given paths, requiring *continuity* of the logarithm. We compute separately the net changes of the imaginary parts of the summands in

$$\log\xi(s) \;=\; -\frac{s}{2}\log\pi + \log\Gamma(\frac{s}{2}) + \log\zeta(s)$$

The net change of imaginary part of the logarithm of $\pi^{-s/2}$ is

$$\mathrm{Im}\big(\log\pi^{-(\frac{1}{2}+iT)/2} - \log\pi^{-2/2}\big) \;=\; \mathrm{Im}\big(-\frac{\frac{1}{2}+iT}{2}\log\pi\big) \;=\; -\frac{T}{2}\log\pi$$

[1] From [Backlund 1914, 1916, 1918]. See also [Titchmarsh/Heath-Brown 1951/1989] pages 212-213. [Backlund 1916] was a thesis done under E. Lindelöf's supervision.

[2] As usual, $\xi(s)$ is the completed zeta function $\xi(s) = \pi^{-s/2}\,\Gamma(\frac{s}{2})\,\zeta(s)$. The usual notation is $S(T) = \frac{1}{\pi}\arg\zeta(s)$, required to be 0 at $s = 2$, and continuous along the vertical line from 2 to $2+iT$ and then to $\frac{1}{2}+iT$. When there is a zero along $(0,1)+iT$, compute $S(T)$ slightly above T.

[3] The $O(1)$ term would also accommodate zeros *finitely-many* other other poles and zeros that conceivably might exist for other zeta-functions and L-functions.

From

$$\log\Gamma(s) \;=\; (s-\tfrac{1}{2})\log s - s + O(1)$$

we have

$$\log\Gamma(\tfrac{s}{2}) \;=\; (\tfrac{s}{2}-\tfrac{1}{2})\log\tfrac{s}{2} - \tfrac{s}{2} + O(1)$$

Thus, the net change from 2 to $\frac{1}{2}+iT$ is

$$\mathrm{Im}\Big(\log\Gamma(\frac{\frac{1}{2}+iT}{2}) - \log\Gamma(\frac{2}{2})\Big) \;=\; \mathrm{Im}\Big((\frac{\frac{1}{2}+iT}{2}-\tfrac{1}{2})\log\frac{\frac{1}{2}+iT}{2} - \frac{\frac{1}{2}+iT}{2}\Big) + O(1)$$

$$= \;\mathrm{Im}\Big((-\frac{1}{4}+\frac{iT}{2})(\frac{\pi i}{2}+\log\frac{T}{2}+O(\tfrac{1}{T}))\Big) - \frac{T}{2} + O(1) \;=\; \frac{T}{2}\log\frac{T}{2} - \frac{T}{2} + O(1)$$

Since $s = 2+i\mathbb{R}$ is within the region of absolute convergence of the Euler product, $\log\zeta(2+it)$ is *bounded* on that line, so the net change in the imaginary part of the argument of $\zeta(s)$ from 2 to $2+iT$ is $O(1)$.

The subtle computation concerns the net change in the argument of $\zeta(s)$ from $2+iT$ to $\frac{1}{2}+iT$. We recall a version of part of a relevant lemma from [Titchmarsh/Heath-Brown 1951/1989], page 213, which uses *Jensen's Lemma* to approximate the number of zeros, hence, the change in argument, in terms of the *growth* of a meromorphic function. We will apply the following lemma to $f(s) = \zeta(s)$:

[0.3] Lemma: Let f be a holomorphic function on a vertical strip $-1 \le \sigma \le 2+1$, except possibly for a simple pole at $s=1$. Suppose that $f(\overline{s}) = \overline{f(s)}$. Assume a lower bound $\mathrm{Re}\, f(2+it) \ge m > 0$, and a family of upper bounds

$$|f(\sigma+it)| \;\le\; M(T) \qquad\qquad (\text{for } \tfrac{1}{4}\le\sigma\le 4 \text{ and } 1\le t\le T)$$

Then, for T not the vertical coordinate of a zero of f, there is the upper bound for change in argument from $2+iT$ to $\frac{1}{2}+iT$

$$\Big|\arg f(\tfrac{1}{2}+iT) - \arg f(2+iT)\Big| \;\le\; \frac{\pi}{\log\big((2-\frac{1}{4})/(2-\frac{1}{2})\big)}\cdot\Big(\log M(T+2) + \log\frac{1}{m}\Big) + \pi$$

[0.4] Remark: Naturally, some of the details are insignificant, being mere artifacts of the proof. At the same time, we give a more specific version of the result than [Titchmarsh/Heath-Brown 1951/1989].

Proof: Let q be the number of vanishings of $\sigma \to \mathrm{Re}\, f(\sigma+iT)$ between $2+iT$ and $\frac{1}{2}+iT$. The vanishing points divide the interval into $q+1$ subintervals on each of which either $\mathrm{Re}\, f \ge 0$ or $\mathrm{Re}\, f \le 0$. In particular, the value of f stays in either the right or left half-plane, so $\arg f$ cannot change more than π in each subinterval. Thus,

$$|\arg f(\tfrac{1}{2}+iT) - \arg f(2+iT)| \;\le\; (q+1)\cdot\pi$$

Using $f(\overline{s}) = \overline{f(s)}$, the count q is the number of zeros of $g(z) = \frac{1}{2}\big(f(z+iT)+f(z-iT)\big)$ on the real interval $\frac{1}{2}\le z\le 2$. Certainly this count is at most the number of zeros of $g(z)$ in the disk $|z-2|\le 2-\frac{1}{2}$.

Let $\nu(r)$ be the number of zeros of g in $|z-2|\le r$. Setting up application of Jensen's lemma,[4] we have an upper bound for q:

$$\int_0^{2-\frac{1}{4}} \frac{\nu(r)}{r}\,dr \;\ge\; \int_{2-\frac{1}{2}}^{2-\frac{1}{4}} \frac{\nu(r)}{r}\,dr \;\ge\; \nu(2-\tfrac{1}{2})\cdot\log\Big(\frac{2-\frac{1}{4}}{2-\frac{1}{2}}\Big) \;\ge\; q\cdot\log\Big(\frac{2-\frac{1}{4}}{2-\frac{1}{2}}\Big)$$

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[4] *Jensen's Lemma* usually appears as follows: for holomorphic f on $|z|\le r$, with no zeros on $|z|=r$, and with $f(0)\ne 0$,

$$\log|f(0)| - \sum_\rho \log\Big|\frac{\rho}{r}\Big| \;=\; \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta \qquad\qquad (\text{summed over zeros } |\rho|<r \text{ of } f)$$

Jensen's lemma leads to an upper bound for the integral:

$$\int_0^{2-\frac{1}{4}} \frac{\nu(r)}{r}\,dr \;=\; \frac{1}{2\pi}\int_0^{2\pi} \log\Big|g(2+(2-\frac{1}{4})e^{i\theta})\Big|\,d\theta \;-\; \log|g(2)| \;\le\; \log M(T+2) + \log\frac{1}{m}$$

giving the lemma. ///

To apply lemma to $f(s)=\zeta(s)$, we show that since $\mathrm{Re}\,\zeta(s)$ has a lower bound $m>0$ on $2+i\mathbb{R}$. Indeed,

$$\mathrm{Re}\,\zeta(2+it) \;\ge\; 1-\sum_{n=2}^{\infty}\frac{1}{n^2} \;>\; 1-\int_1^\infty \frac{dx}{x^2} \;=\; 1-\frac{1}{2-1} \;=\; 0$$

Because of the strict inequality, there is a strictly positive lower bound m.

On vertical lines, $\zeta(\sigma+it)$ has a polynomial bound, from the functional equation, and from Phragmén-Lindelöf. Thus, the net change in the argument of $\zeta(s)$ from $2+iT$ to $\frac{1}{2}+iT$ is $O(\log T)$.

Thus, altogether, the argument principle gives

$$N(T) \;=\; \tfrac{1}{2}\cdot\frac{1}{2\pi}\cdot 4\cdot\Big(\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}-\frac{T}{2}\log\pi\Big)+O(\log T)$$

$$= \frac{1}{\pi}\cdot\Big(\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}(1+\log\pi)\Big)+O(\log T) \;=\; \frac{T\log T}{2\pi}-\frac{T}{2\pi}\log 2-\frac{T}{2\pi}(1+\log\pi)+O(\log T)$$

$$= \frac{1}{2\pi}\cdot T\log T \;-\; \frac{\log 2\pi e}{2\pi}\cdot T \;+\; O(\log T)$$

which is the asserted asymptotic. ///

Bibliography:

[Backlund 1914] R.J. Backlund, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, C.R. **158** (1914), 1979-81.

[Backlund 1916] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916.

[Backlund 1918] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Math. **41** (1918), 345-75.

[Titchmarsh/Heath-Brown 1951/1989] E.C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford University Press, 1951. Second edition, revised by D.R. Heath-Brown, 1986.

Letting $\nu(t)$ be the number of zeros of size less than t,

$$-\sum_\rho \log\Big|\frac{\rho}{r}\Big| \;=\; \sum_\rho(\log r-\log|\rho|) \;=\; \sum_\rho\int_{|\rho|}^r \frac{dt}{t} \;=\; \int_0^r \nu(t)\,\frac{dt}{t}$$

Thus,

$$\int_0^r \frac{\nu(t)}{t}\,dt \;=\; -\sum_{|\rho|<r}\log\Big|\frac{\rho}{r}\Big| - \log|f(0)| \;=\; \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta \;-\; \log|f(0)|$$

This applies to circles centered at points other than 0, by the obvious translation.