Math 8701 / Dec 08, '21

$\exists$ exam 4, Fri,

entirely virtual: I'll email you a PDF...

(postponing is fine — let me know —)

* A little more about l.f.t.'s :

Thm: Given z_1, z_2, z_3 distinct $\in \mathbb{C}$ ~~(or $\mathbb{P}^1$)~~

$\exists!\, g \in GL_2(\mathbb{C})/\text{center}$ (= scalars) s.t.

$$g(z_1) = 0, \ g(z_2) = 1, \ \& \ g(z_3) = \infty$$

("the cross-ratio" —)

Pf (nevermind uniqueness —)

Steps: $z_1 \xrightarrow{\checkmark} 0$ & $z_3 \to \infty$ ly

l.f.t. $z \longrightarrow \dfrac{z - z_1}{z - z_3}$ $\quad (z_1 \neq z_3)$

* "Can ~~we~~ we want to send the image of z_2 to 1 ?

WHILE not moving 0 or ∞ !

(The subgp of $GL_2(\mathbb{C})$ fixing 0 & fixing ∞ is $\left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$ ✓ ☺)

$$\equiv \quad z \longrightarrow \alpha z, \quad \alpha \in \mathbb{C}^\times$$ ✓ ☺

$\Longrightarrow$ further compose: with $\underbrace{z \longrightarrow \frac{1}{\text{prev. img of } z_2} \cdot z}_{\text{fix } 0,\infty, \text{ \& prev img}(z) \text{ to } 1\,!}$

So whole is $$z \longrightarrow \underbrace{\dfrac{\;\dfrac{z - z_1}{z - z_3}\;}{\;\dfrac{z_2 - z_1}{z_2 - z_3}\;}}_{\text{So } \exists \geq 1}$$ ✓ ☺

$=$ "the cross-ratio"

◎ Conformal Mapping: (~ Riemann Map thm ✓ inexplicit ~)

If $\exists$ biholo $f : U_1 \to U_2$ (open in $\mathbb{C}$),

then $\{\text{holo fcn on } U_1\} \approx \{\text{holo fcn on } U_2\}$ by comp. w/ f ✓

"Bigons" = "Bi-gons" ($\neq$ Byegones —)

Thm: essentially* all bigons are isomorphic

($\exists$ holo bijection one $\to$ another)

Open

Content/explain: Has 2 sides, arcs of $\odot$'s, or straight lines (circles through ∞)

non-deg.

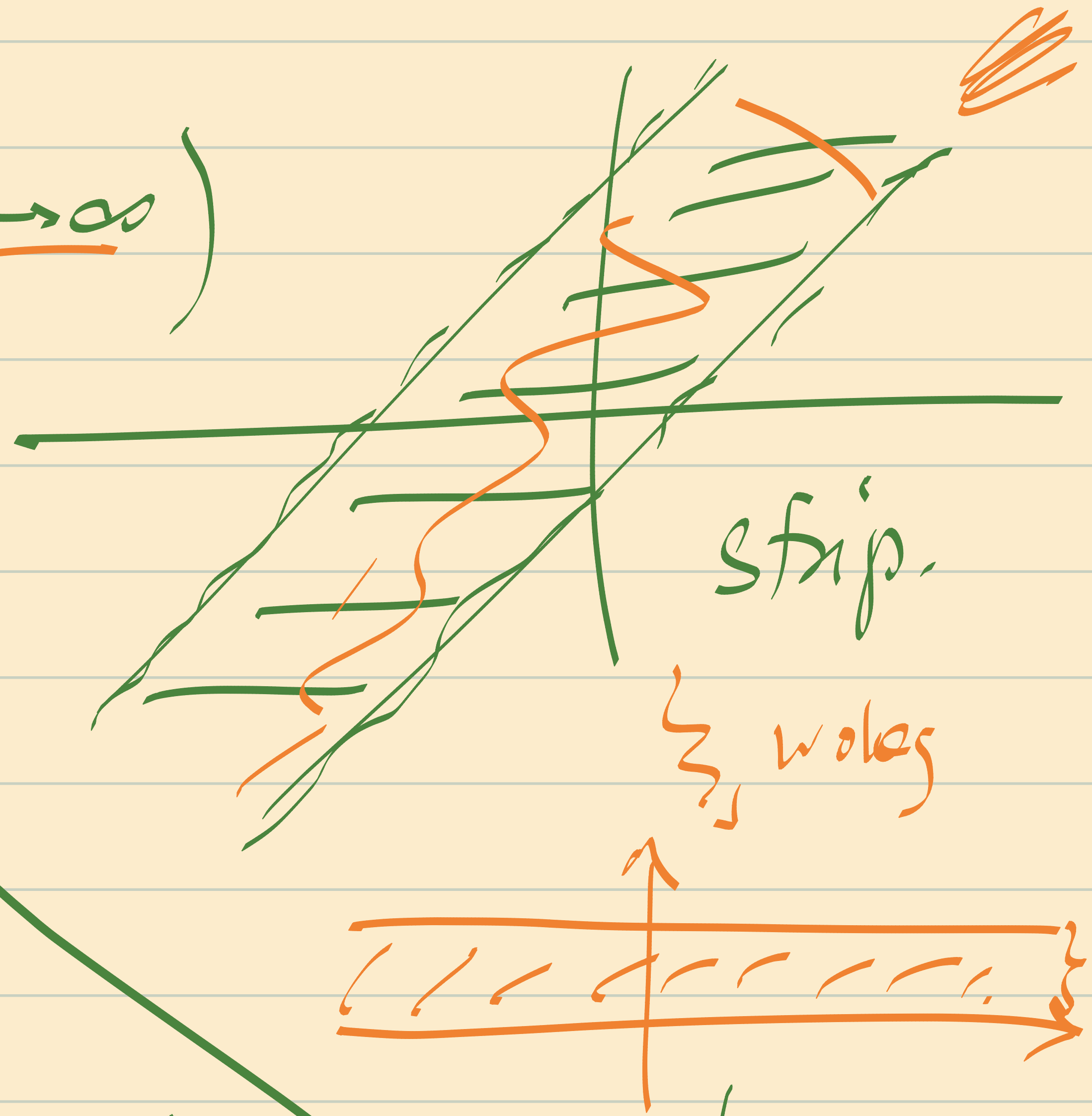
extreme

(other vertex $= \infty$)

sectors

+ degenerate: 2 vertices are the same pt

 & ($\# \to \infty$)



Precisification of thm:

non-deg ($\equiv$ two vertices are not the same pt)

are all isomorphic (= biholomorphic)

& all "degen." (vertices = same pt) are biholo

For non-deg:

Send one vertex to ∞

one to 0

(by l.f.t.: nice on $\mathbb{P}^1$)

invertible ✓ :)

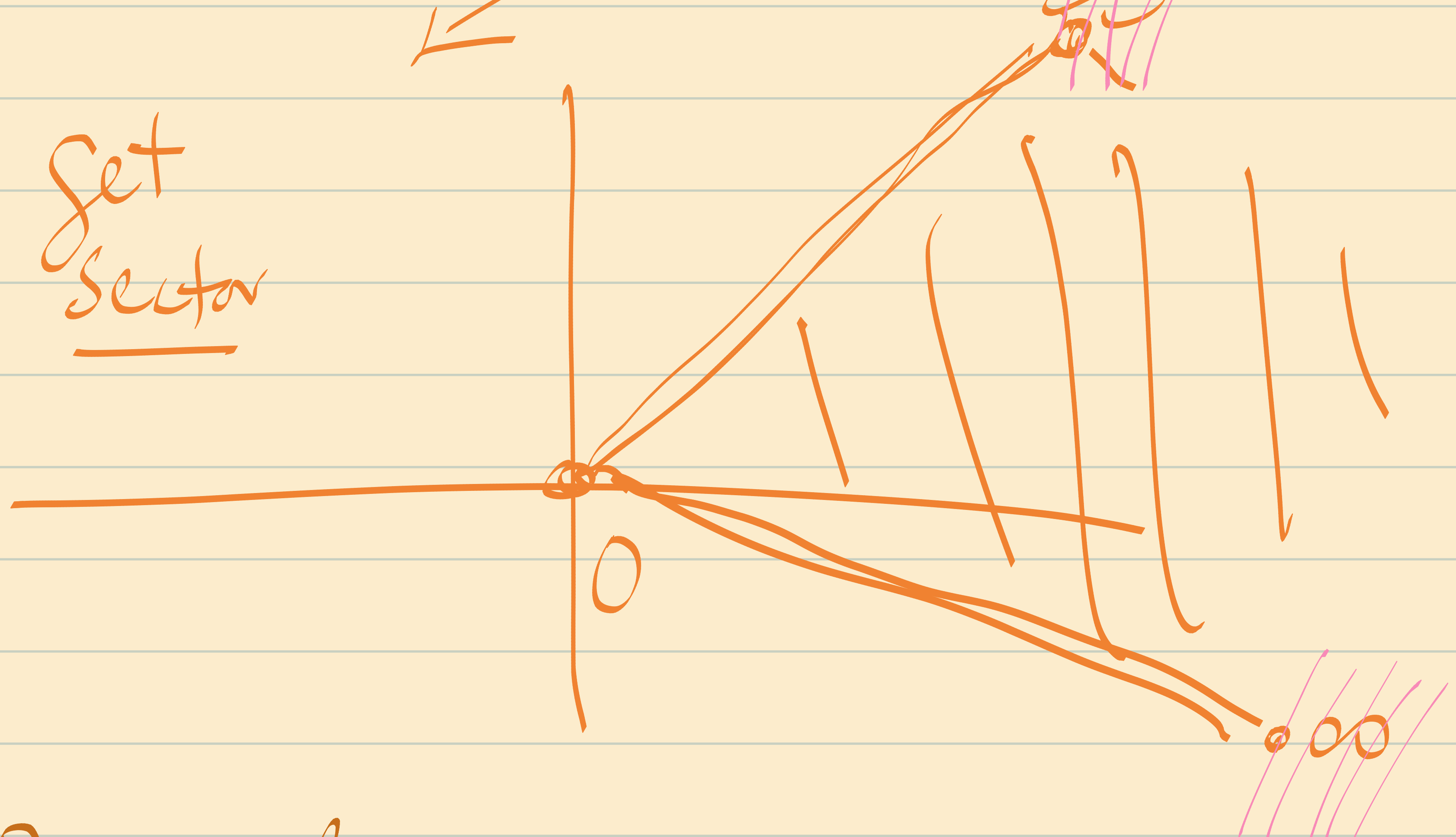
→ Angle-preserving + $\begin{matrix}\text{lines}\\+\\\text{circles}\end{matrix}$-preserving

conformal ✓

of l.f.t.'s

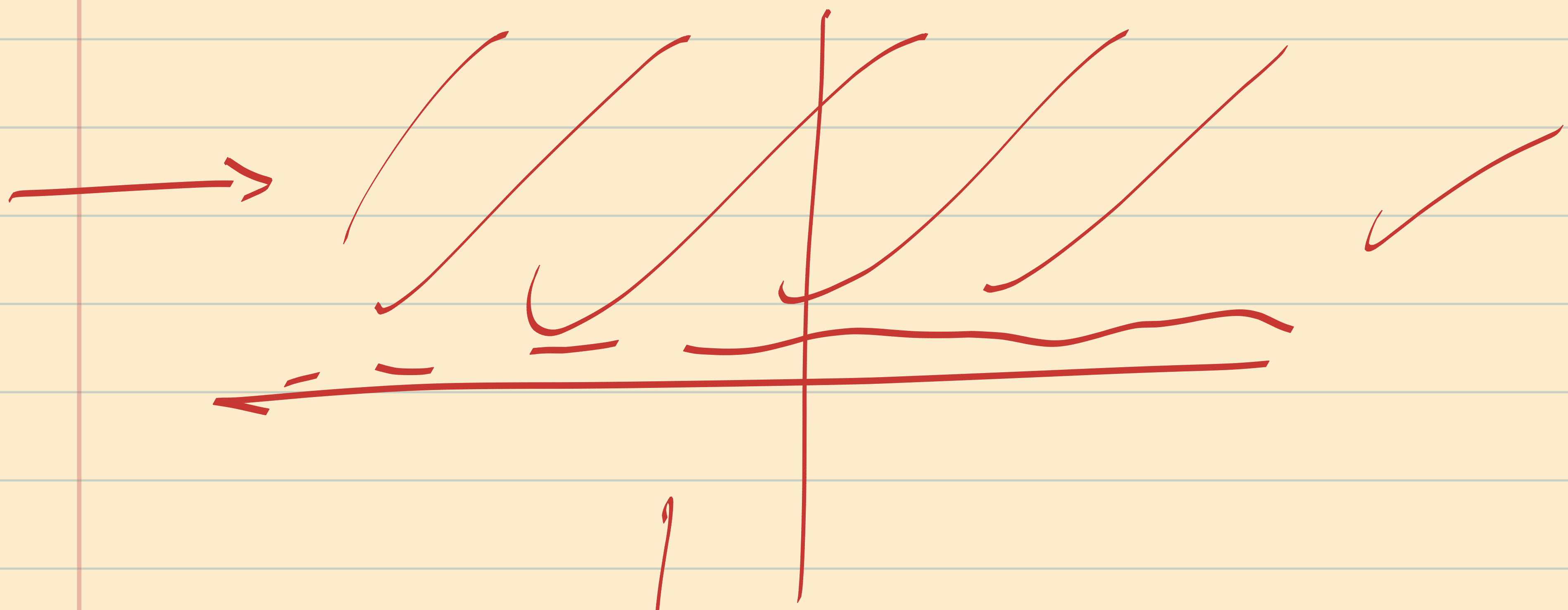
?

Set Sector

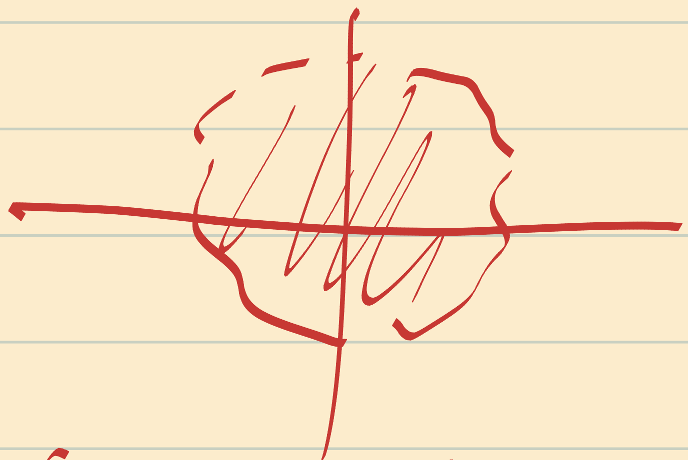


Rotate by $z \longrightarrow \mu \cdot z$ with $|\mu| = 1$:

$\boxed{\text{Take } \tfrac{1}{2\alpha}\text{th power to get half-plane!}}$



Inverse Cayley



Hence (non-deg) bigon $\xrightarrow{\cong}$ { } $\xleftarrow{\cong}$ every other, also

$\exists$ $\exists$

Deg?

map vertex to ∞ by l.f.t

lines-and-circles presentation:

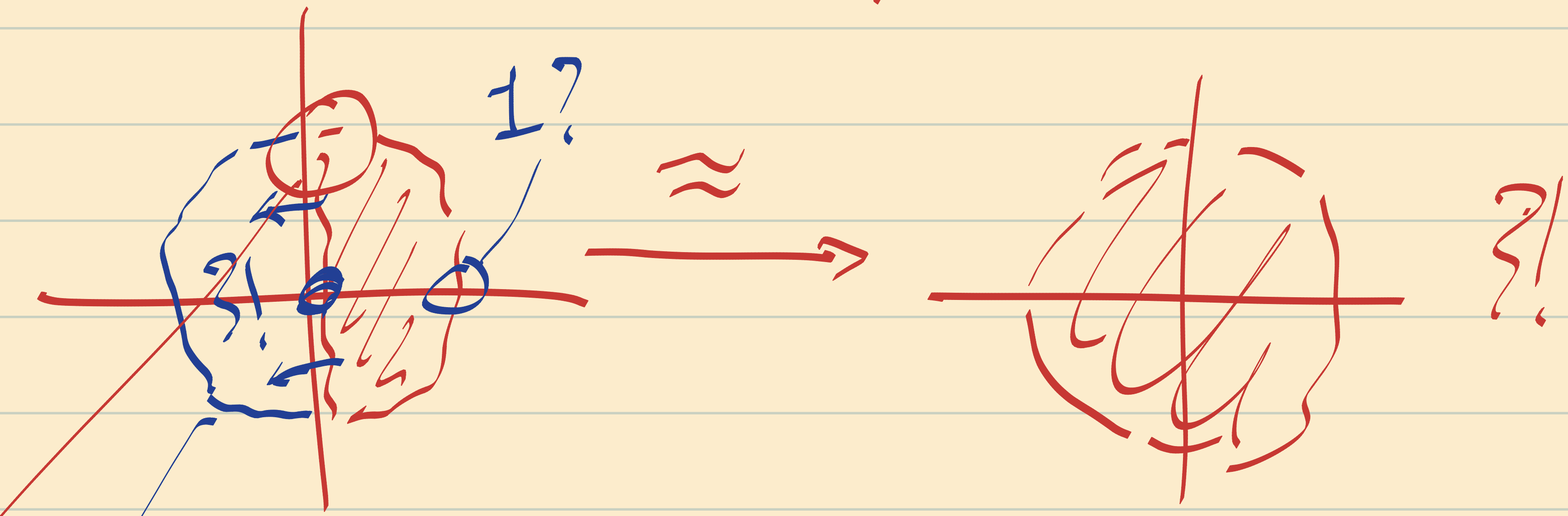
2 arcs-of-circles-or-lines meeting only at ∞

⌊ through ∞ are straight lines

& cond'n that 2 lines meet only @ ∞ ⟹ parallel :-) → rotate & dilate to normalise ← :-)

Concrete examples: (w/o context – tricky?!)

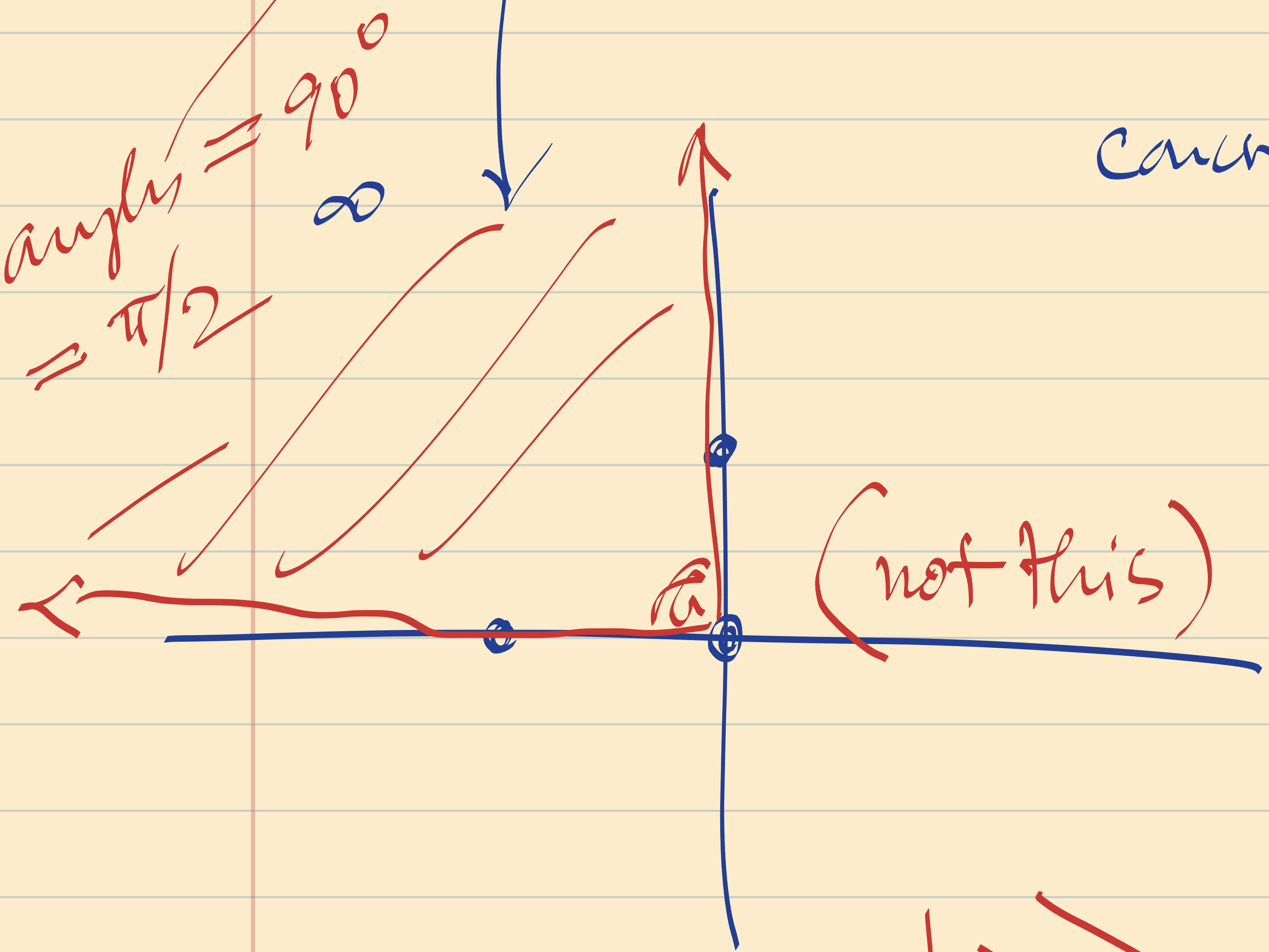
Eg.:



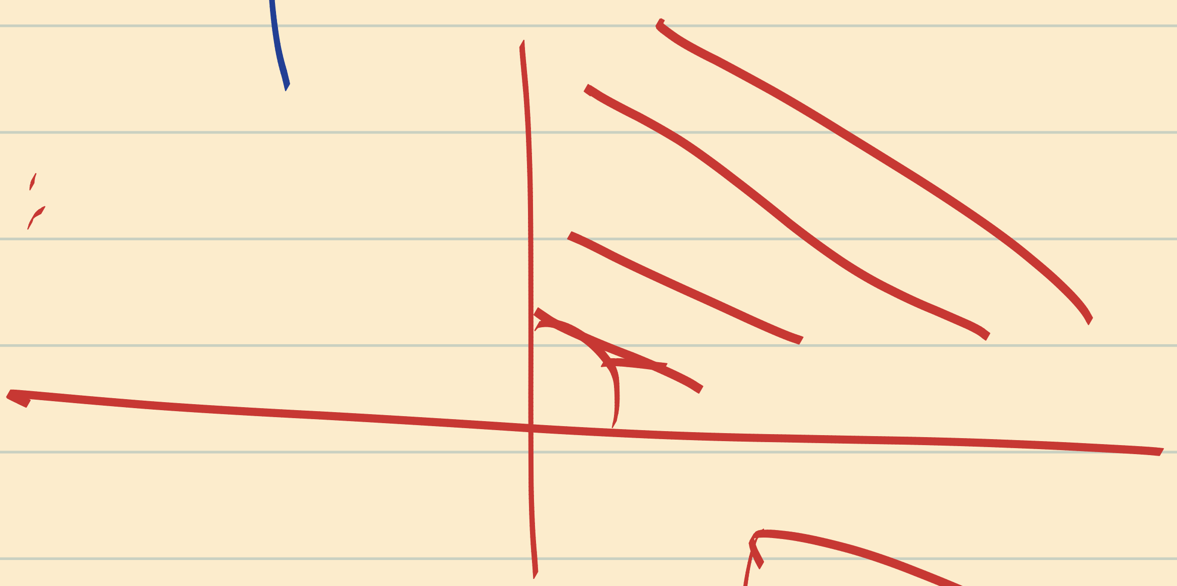
Send i to ∞ & $-i$ to 0: $z \longrightarrow \dfrac{z+i}{z-i}$ ✓

concrete choice: $1 \longrightarrow \dfrac{1+i}{1-i} = i$ (!?) ✓

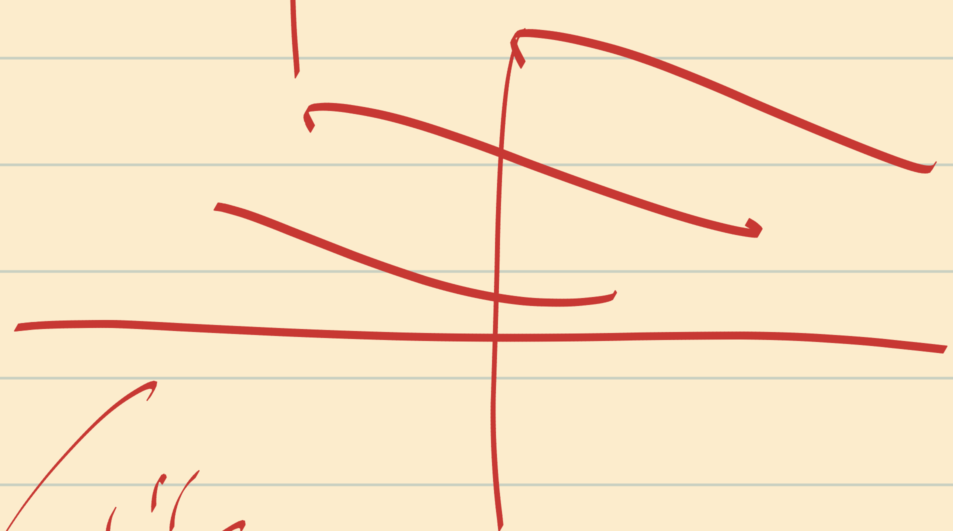
& $0 \longrightarrow -1$



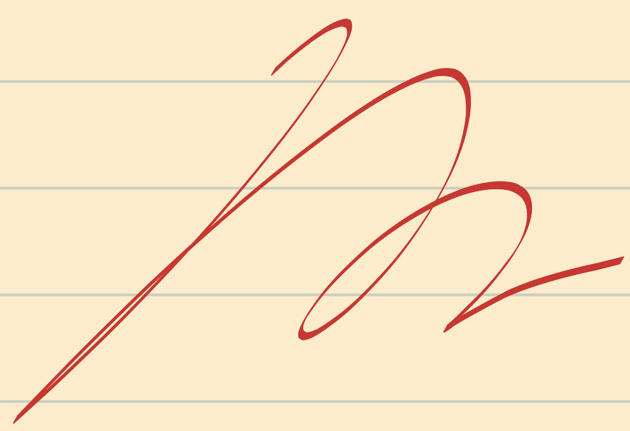
$\times\, i^{-1}$:

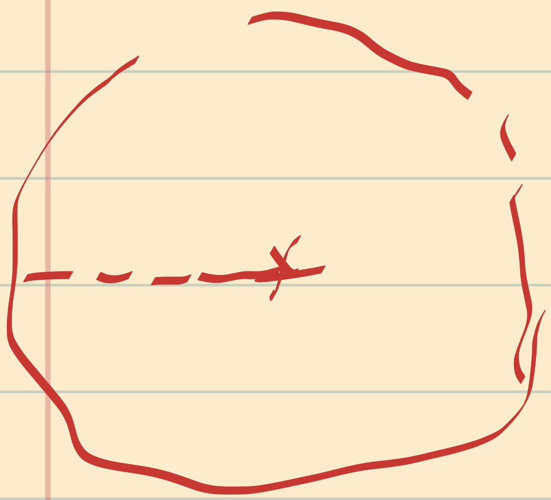
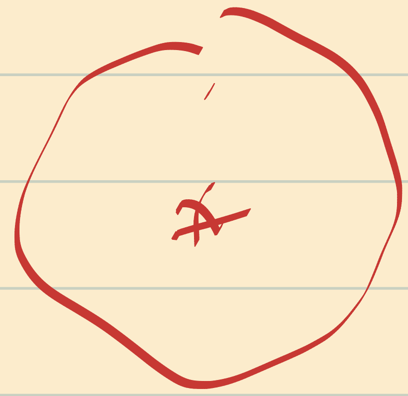


square:

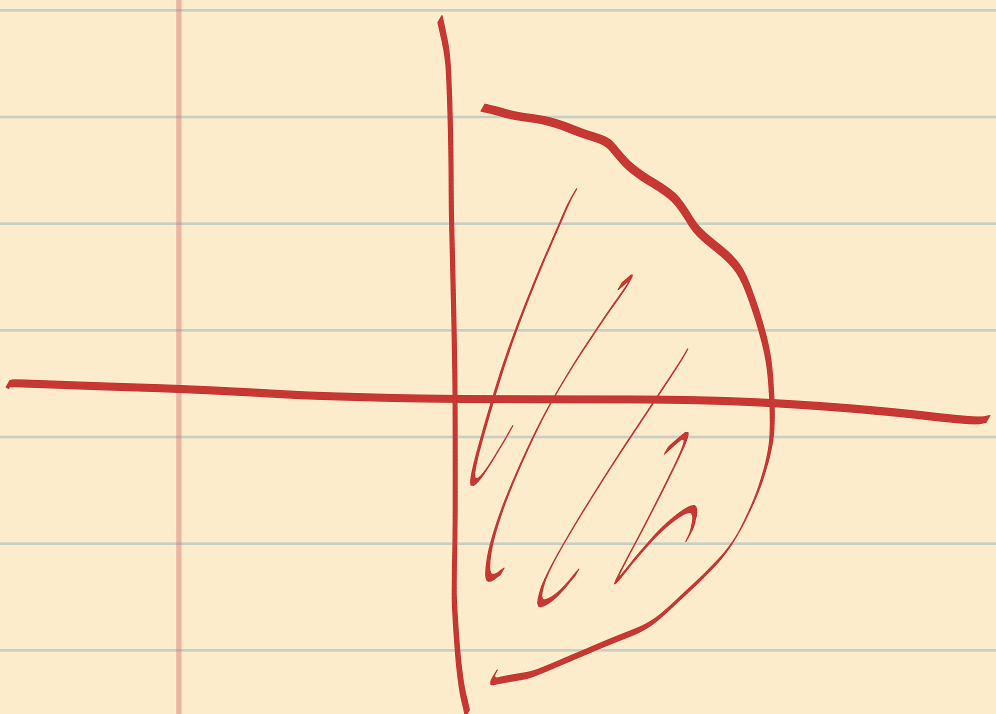


Inverse Cayley ✓



 $\xrightarrow{!?!}$  :

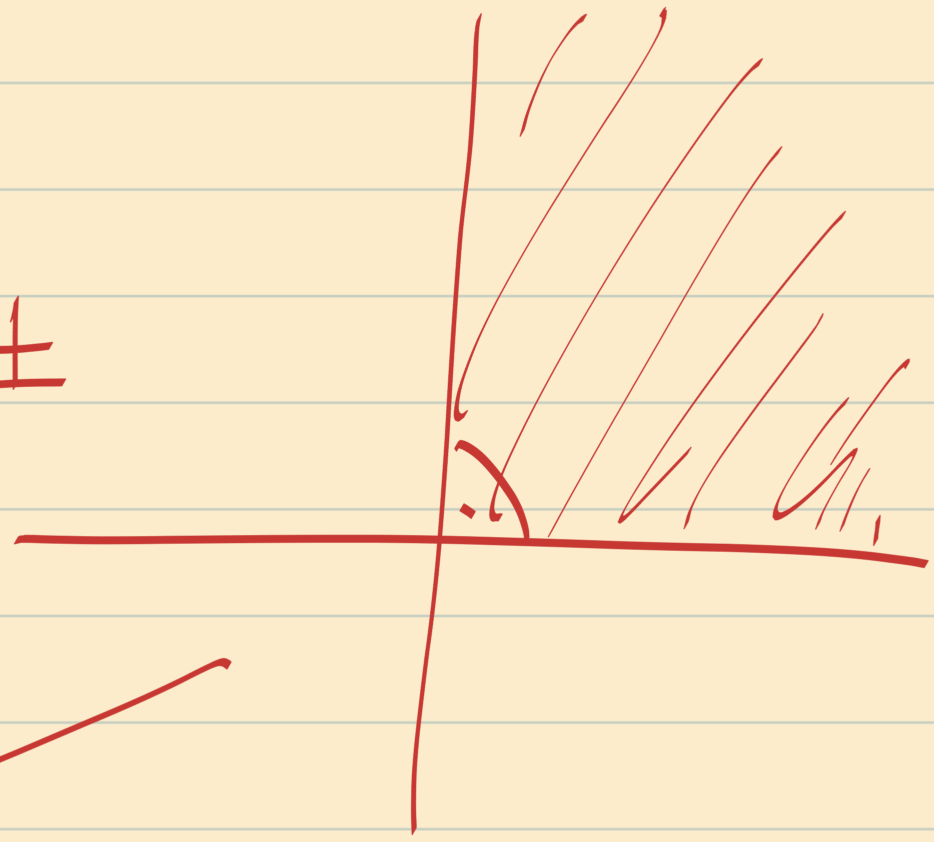
$\downarrow$ std $\sqrt{\ }$



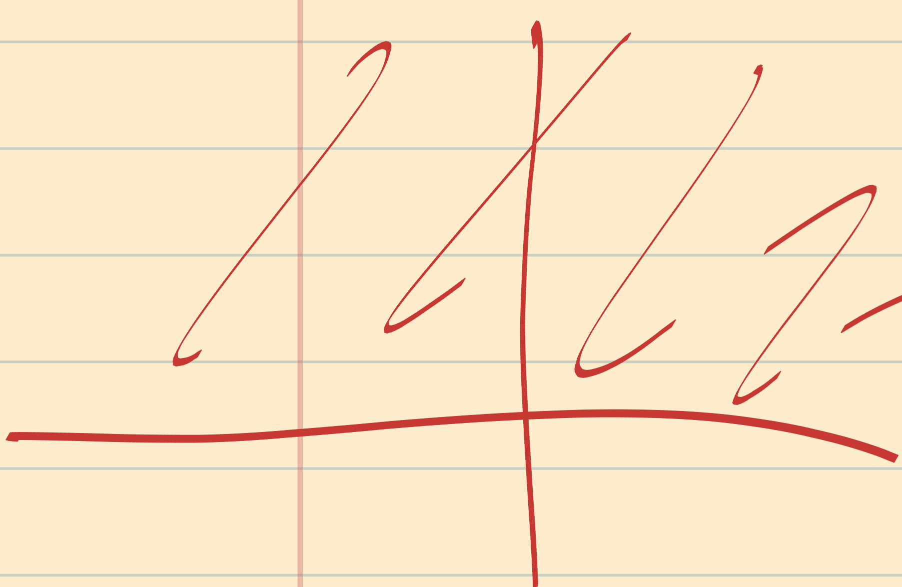
$+i \to \infty$

$-i \to 0$

$\longrightarrow$ $\pm$



square:

 $\xrightarrow{\text{inverse Cayley}}$  ✓

Exam remote : I'll send you email ~ 2:30
(or your choice)

— or delay till Monday —

Bonus!

Are there entire fcns f with $f(x \in \mathbb{R}) \downarrow\downarrow 0$

& $\int_{\mathbb{R}} x^n f(x)\, dx = 0$ for $n = 0, 1, 2, \ldots$

lots!? ?

Answer: $\exists$ many such f, but you'll not be able to express them in elementary terms !?! ☺

Eg., certain f's in Paley-Wiener space succeed

$\Downarrow$

"monsters" — F.T.'s of $C_c^\infty(\mathbb{R})$ $\equiv$ not really elementary

$(\sim x^{-1/x} \sim 2! \sim)$

Eg., via F.T.

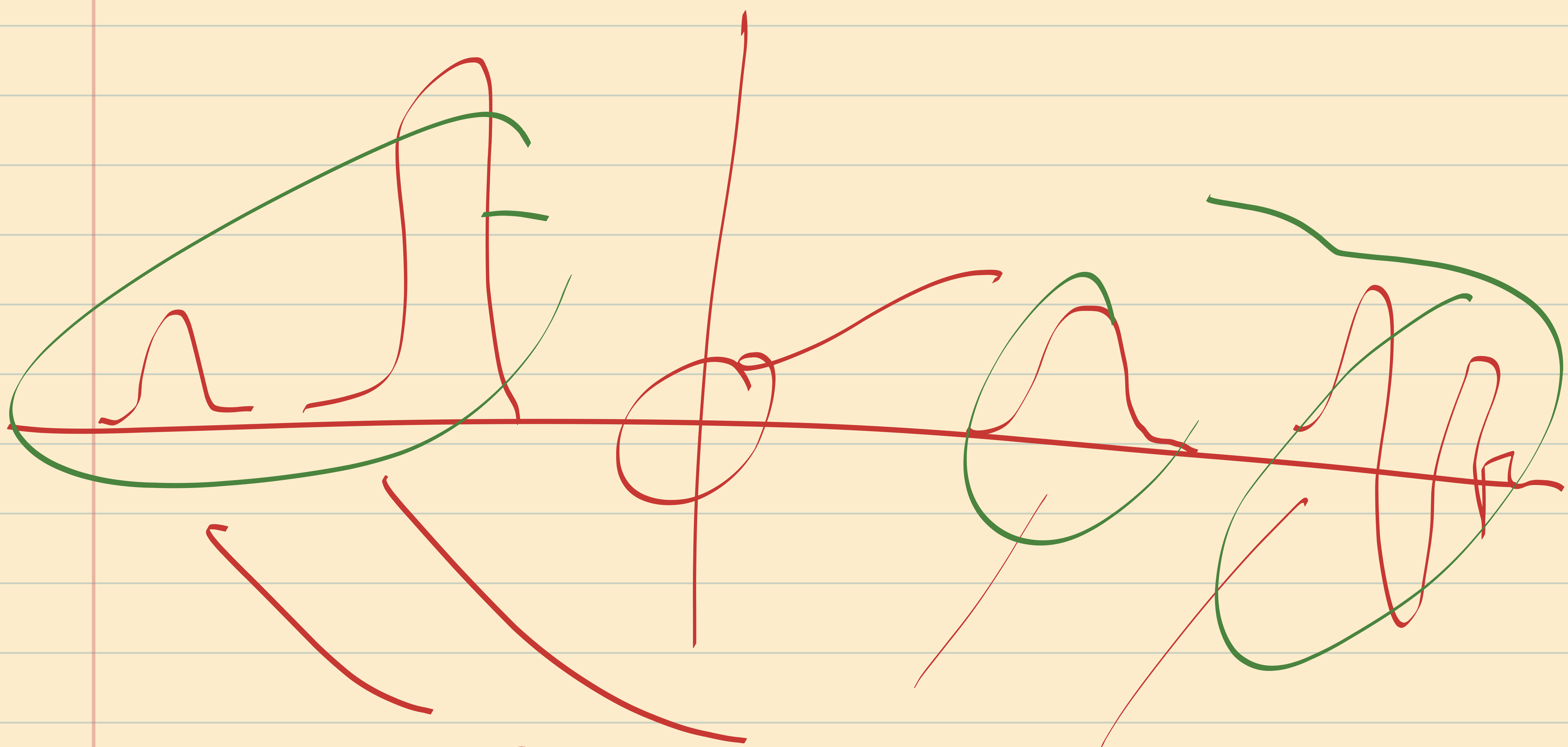
$$\int x^n \cdot f = 0$$

$$\Updownarrow$$

$$\hat{f}^{(n)}(0) = 0 \quad \forall n \, ;$$

F.T. converts $\times$ by x^n to $c \times$ diff'n by $\left(\frac{d}{dx}\right)^n$

easy to fulfill: pictorially:



such f are not "accessible"

not formulaic — F.T. does simplify into anything

(see proof of P.W. thm)