19 Jan 2022

- Put on-line these notes — (no audio)
- $\exists$ hmwk/examples Jan 28 (exam Feb 4 (?)) Fri
  - explicit Σ's & prod's ~ sine, et al
  - partial frac. — — —
  - intro to harmonic fcns

  $\exists$ notes

coincidence?!

are f on $\mathbb{R}^n$, st $\Delta f = 0$

$$\nabla \qquad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \begin{cases} \text{rot-inv't} \\ \text{transl'n-inv't} \end{cases}$$

const-coef

$\text{Re} \log f$ harmonic ...!

counting 0's (Jensen's fml)

counting 0's of $\zeta(s)$ ...! :) — 1858

past 1900

See notes for historical: Euler & Basel

$\left(\zeta(2) = \pi^2/6 \right)$

use * also

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

is 0 @ $z = \pm n$

(cvgt!?)

0's @ all $\neq 0$ integers

Proof? 3 steps ...

① Partial frac. of $\dfrac{1}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$!?! !?

"$(\sin \pi z)^2$"

not $\sin \pi z^2$...

unif pointwise cvgt on cpts

DOUBLE poles/0's

$\not\ni$ any $\ell \in \mathbb{Z}$

cvge

Game/method: show $\dfrac{\pi^2}{\sin^2 \pi z} - \sum_{n \in \mathbb{Z}} \dfrac{1}{(z-n)^2}$?!

is bdd & entire, so (Liouville) is const

& const. $= 0$ — ✓

— (as $\operatorname{Im} z \to \pm\infty$);

$\boxed{\text{Away from } \mathbb{R}}$, both sides are hol —✓

& RHS (elem) $\downarrow 0$ like $\dfrac{1}{|z|}$ or $\dfrac{1}{|\operatorname{Im} z|}$.

LHS $\sim \dfrac{1}{\sinh^{(2)}(\operatorname{Im} z)}$ = exp decr. in $\operatorname{Im} z$

(?)

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\boxed{|\operatorname{Im}(z)| \leq 1}$: poles: claimed to cancel;

(*) Both LHS & RHS are periodic, w/

period 1 (either $(z+1)$ = either(z) —

either$(z+\ell)$ = either(z) —)

$\forall \ell \in \mathbb{Z}$

?, LHS ✓ RHS: it is an avg/sum

of integer translates of $\frac{1}{z^2}$ ⟶

$$\mathrm{RHS}(z+\ell) = \sum_{n\in\mathbb{Z}} \frac{1}{((z+\ell)-n)^2}$$

z is arb + abs. cvge

$$= \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} = \text{same}$$

n by $n+\ell$

z is so

periodic

suffices to compare polar parts @ $z=0$:

LHS near $z=0$: $\dfrac{\pi^{2}}{\sin^2 \pi z} =$

$$= \frac{\pi^2}{\left(\pi z - \frac{(\pi z)^3}{3!} + \dots\right)^2} = \frac{\pi^2}{\pi^2 z^2 + \text{h.o.t.}}$$

p.s. sine @ 0

$$\overset{\text{sem}}{=} \frac{1}{z^2}\left(\frac{1}{1+\text{h.o.t.}}\right)$$

$$= \frac{1}{z^2}\left(1 - (\text{h.o.t.}) + (\text{h.o.t.})^2 - \dots\right)$$

$$= \frac{1}{z^2} + \text{holo}$$

RHS: $\underbrace{\frac{1}{z^2}}_{\text{polar part}} + \underbrace{\sum_{n\neq 0} \frac{1}{(z-n)^2}}_{\text{cvges well: hol. near } 0}$ ✓ :)

polar part: $\equiv$ same as that of $\frac{\pi^2}{\sin^2 \pi z}$ ✓ :)

$$\frac{\pi^2}{\sin^2 \pi z} - \sum \frac{1}{(z-n)^2} = \underline{\text{entire}}$$ ✓

sing. @ $\mathbb{Z}$ are "removable"

Liouville?! bdd!?

Bdd in $|\operatorname{Im} z| \geq 1$ ... ✓

In $|\operatorname{Im} z| \leq 1$? ⊛ periodic, so

suffices to prove bdd in

$\{|\operatorname{Im} z| \leq 1 \;\&\; 0 \leq \operatorname{Re} z \leq 1\}$ = cpt

& LHS − RHS = cont., so bdd ⊛

$\Longrightarrow$ LHS − RHS = bdd entire $\Longrightarrow$ const. $= 0$ ✓

Further, as $\operatorname{Im} z \to \pm\infty$: LHS − RHS $\to 0$, ditto ☺

Next step!

$$\pi \cot \pi z \overset{\text{claim}}{=} \frac{1}{z} + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{z-n} + \frac{1}{z+n} \right)}_{\text{together for conv'ge:}}$$

!?

π's ?!

$\cancel{\sum_{n\in\mathbb{Z}} \frac{1}{z-n}}$ problems ☹

this is NOT quite conv'd version of $\frac{1}{z-n}$ nor $\frac{1}{z-n} + \frac{1}{z+n}$

periodic ???

RHS

Start:

$$\underbrace{\frac{\pi^2}{\sin^2 \pi z}} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \quad ✓$$

$$\underset{\text{regroup}}{=} \frac{1}{z^2} + \sum_{n=1}^{\infty}\left(\frac{1}{(z-n)^2}+\frac{1}{(z+n)^2}\right)$$

✓

(you check) ! ✓ :)

~~-1 on RHS~~

$$\frac{d}{dz}\left(\cancel{-}\pi\cot\pi z\right)$$

Int. termwise:

$$\pi\cot\pi z \overset{?}{=} C + \frac{1}{z} + \overbrace{\underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)}_{\sim 1/n^2}}^{\text{still abs. cvgt.}}$$

! all but C are odd $(z \to -z)$

so $C = 0$ ✓ :)

3rd $\quad \frac{\pi}{z} \cot \pi z = \log(\sin \pi z)$ (!?!) — ☺

$$\frac{d}{dz}\left(\log \sin \pi z\right) = \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{(z-n)} + \frac{1}{(z+n)}\right)$$

(algebra)
integrate & exp

$$\sin \pi z = e^{\exists C} \cdot z \cdot \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

Look @ leading coef: $e^C = 1$,

(Cor $\zeta(2) = \frac{\pi^2}{6}$, etc.)