($\exists$ examples / hmwk 05 ... due ~~exam 05~~ 21 Jan 2022
7 days from now

exam 05 14 days " ")

"Partial fractions" → $\boxed{\text{elem \& alg.}}$

→ obv. ∞-sum extn

→ amazing thm!

$\underline{\underline{\text{Alg.}}}$: $\underline{\underline{\text{"know"}}}$ $\dfrac{P(z)}{Q(z)}$ $(P, Q \in \mathbb{C}[x])$

$$= \sum_{\substack{\cancel{\text{roots}} \text{ of } Q \\ 0\text{'s } z_0}} \left(\frac{c_1}{z-z_0} + \frac{c_2}{(z-z_0)^2} + \dots + \frac{c_n}{(z-z_0)^n} \right)$$

order of the 0

$+$ poly

Reason $=$ (field $[x]$ = Euclidean ring & UFD, PID ...)

Not provable ?! to calc. I students ? !

Extreme case: $\dfrac{1}{(z-\lambda_1)\cdots(z-\lambda_n)}$ distinct λ_j's

$= f(z)$

$$\underset{\substack{\text{can}\\ \text{verify}}}{\overset{\exists\,\text{fml!}}{=}} \sum_{j=1}^{n} \frac{1/f'(\lambda_j)}{z-\lambda_j}$$

($\exists$ analogues more generally)

Proof general case: suffices $\dfrac{1}{A(z)\cdot B(z)}$

$$= \frac{\exists\, P(z)}{A(z)} + \frac{\exists\, Q(z)}{B(z)}$$ when $\gcd(A,B)=1$ as polys

($\exists$ Euclidean alg. for polys over field)

Key point: given A, B coprime, $\exists\, C, D$ st $A\cdot C + B\cdot D = 1$ (Bezout's thm) (True in any ~~Euclidean~~ PID)

So, w/ $\quad A\cdot C + B\cdot D = 1$

$\div AB$: $\quad \dfrac{C}{B} + \dfrac{D}{A} = \dfrac{1}{AB}$

as desired ✓ :-)

$\approx$ writing fractions $\frac{p}{q} \in \mathbb{Q}$ as Σ's of $\frac{\text{int}}{\text{primes}^e}$

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$\sim$ Babylonian fraction arithmetic ?!

<u>Limiting case</u>: ∞-many poles of mero f on $\mathbb{C}$

(above = rational, $<\infty$ ∞'s —)

At pole z_0 ?!

$\exists$ Laurent exp'n, & "mero" (~~ess sing~~)

$\exists <\infty$ neg. terms ✓ :-)

"The $\boxed{\text{polar part}}$ @ z_0" of mero fcn f near z_0 is neg-term part of Laurent expn of f @ z_0:

$$\underset{\exists \text{ punc'd disk } 0<|z-z_0|<\varepsilon}{f(z)} = \underbrace{\frac{c_{-n}}{(z-z_0)^n} + \cdots + \frac{c_{-1}}{(z-z_0)}}_{} + \underbrace{\text{hol}}_{\substack{\text{p.s.} \\ \text{conv'gt} \\ \text{in} \\ \text{disk}}}$$

z

Converse?

Ex: make some fcns w/ ∞-many poles:

have poles @ $0, -1, -2, -3, \ldots$?!

$$\sum_{n=0}^{\infty} \frac{?}{z+n}$$

choose to make Σ cvgt ...

Can't be just "1"

Recall (!?)

partial frac'n ☺ cycle

must add to get $\Gamma(z)$

$$\underset{z}{\Gamma(s)} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n/n!}{z+n}}_{\text{poles! ☺}} + \underbrace{\int_1^{\infty} t^z e^{-t} \frac{dt}{t}}_{\text{entire}}$$

Amazing Thm: $\forall$ discrete subset Z of $\mathbb{C}$,

$\forall$ specified polar data $\underbrace{\mathrm{poly}_{z_0}\left(\frac{1}{z-z_0}\right)}_{\text{finite "nose" of Laurent}}$

$\exists$ mero fcn f on $\mathbb{C}$ s.t. polar part of f @ every $z_0 \in Z$ is $\mathrm{poly}_{z_0}\left(\frac{1}{z-z_0}\right)$!

Amazing: discrete subsets can be bunchy,

eg., (hmmh) $Z = \{\log n : n = 1, 2, \dots\}$

& then $\displaystyle\sum_{n=1}^{\infty} \frac{1}{z - \log n}$ not cvgt

Nonetheless, thm $\Rightarrow$ $\exists$ mero f on $\mathbb{C}$ w/ (polar part $\frac{1}{z-\log n}$ at all $\log n$'s)

precludes adjusting numerator! :) !?!

???

pf: some cleverness :) See notes :)

Aux: "discrete" $\equiv$ $\forall z \in Z$, $\exists$ open disc D
$Z \subseteq \mathbb{C}$ centered at z st

$$D \cap Z = \{z\}$$

Ex: Any discrete subset of $\mathbb{C}$ is countable

Related: For $\{r_\alpha : \alpha \in A\}$ of positive reals r_α

has "$\sum_{\alpha \in A} r_\alpha$" $< \infty \implies$ A ctble ~ ex!

↑ ?!

$$\underset{\text{def'd}}{=} \sup_{\substack{\text{finite} \\ A_0 \subseteq A}} \left(\sum_{\alpha \in A_0} r_\alpha \right) \qquad \left(\overset{\text{ex}}{\iff} \text{abs cvgce} \right)$$

:)

Grant latter: to prove $\boxed{\text{discrete set } Z \text{ is ctble}}$:

$$Z = \overset{\infty}{\underset{n=1}{\bigcup}} \overbrace{\big(Z \cap \underbrace{\{z : |z| \le n\}}_{\text{finite measure}} \big)}^{Z_n}$$

Suff. to show Z_n ctble ✓ :-)

Put open disk (D_z) of rad R_z around each $z \in Z_n$

st $D_z \cap Z = \{z\}$ — discreteness

Then (!) area of $\underbrace{\{z : |z| \le n\}}_{\text{finite}} \ge \sum_{z \in Z_n}' \text{area of } \underbrace{D_z}_{\text{disjt}}$

finite $\Longrightarrow$ ctble ✓

:-)

✓ Partial frac.? ✓ :-)

Intro harmonic fcns

on $\mathbb{R}^n$, w/ $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ "Laplacian"

$f \in C^2$ $\equiv$ $\underbrace{\text{cont. twice diff}}_{\text{needlessly strong}}$: assume that $f, \frac{\partial f}{\partial x_j}, \frac{\partial^2 f}{\partial x_j^2}$ have sensible pointwise values)

& $\Delta f = 0$

Invce/eqce : transl'n eqt ($\Leftarrow$ const coef) :

$$\Delta(f(x+x_0)) = (\Delta f)(x+x_0)$$

rot'n-invce

$$\Delta(f(Rx)) \stackrel{?}{=} (\Delta f)(Rx)$$

ex $\rightarrow$ R = rot'n $\in SO(n,\mathbb{R})$, $R^T R = 1_n$

Harmonic fcns help w/ holo :

either $\log(\mathrm{Re}\ \text{holo}) =$ harmonic

or $\cancel{\mathrm{Re}(\log \text{holo})} =$ "

where holo $\neq 0$

Which? : observe:

$\partial \quad \bar\partial$

$$\boxed{\Delta = 4 \cdot \frac{\partial}{\partial z} \circ \frac{\partial}{\partial \bar z}}$$

on $\mathbb{C} = \mathbb{R}^2$

$$= 4 \cdot \frac{\partial}{\partial \bar z} \circ \frac{\partial}{\partial z}$$

$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$?

opp. sign.

$$\Delta(\log|f|) = \Delta\left(\frac{1}{2}\left(\log f + \log \bar f\right)\right)$$

$$= 2\,\partial\bar\partial\,(\underbrace{\log f}_{\exists\ \text{holo}} + \underbrace{\log \bar f}_{\exists\ \text{anti-holo}})$$

near where $f \neq 0$

$\bar\partial(\text{that}) = 0 \qquad \partial(\text{that}) = 0$

$= 0$ ✓ ☺

Next time: Mean value prop & Poisson $\int$-fml (~ Cauchy's eq. for holo)

$\mathbb{R}^n$: f harmonic on closed ball B_r

then $f(\text{center})$ = avg of values of f on boundary

+ classify harm. fcns on annuli

~ Laurent for holo