Examples 05 this Fri
Exam 05 next Fri

24 June 2022

Harmonic fcns $\leftrightarrows$ holo & Laplacian Δ
in 2D
on $\mathbb{R}^n$ $\quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$

On $\mathbb{R}^2 \simeq \mathbb{C}$! $\Delta = 4 \cdot \partial \circ \bar{\partial}$
$= 4 \cdot \bar{\partial} \circ \partial$

$\bar{\partial} = \frac{\partial}{\partial \bar{z}}$, $\partial = \frac{\partial}{\partial z}$

$$\boxed{\text{! On } \mathbb{R}^n \text{, all transl-inv't (const. coef), rot-inv't diff ops are polys in } \Delta}$$

Laplace's eq'n $\Delta u = 0$ or $\Delta u = f$
elliptic
u harmonic
given

✓ Heat eq'n $\left(\Delta_x - \frac{\partial}{\partial t}\right) u = 0$ (& init. cond'n)
parabolic

(linear) Wave eq'n $\left(\Delta_x - \frac{\partial^2}{\partial t^2}\right) u = 0$ (etc.)
hyperbolic

Mean value property: (holds in $\mathbb{R}^n$!)

e.g., $\mathbb{R}^2 \simeq \mathbb{C}$

Thm: u holo on closed disk & exts cont. to boundary, then

$$u(\text{center}) = \text{avg of } u \text{ over boundary!}$$

In $\mathbb{C} \simeq \mathbb{R}^2$, that avg-over-boundary can be written out:

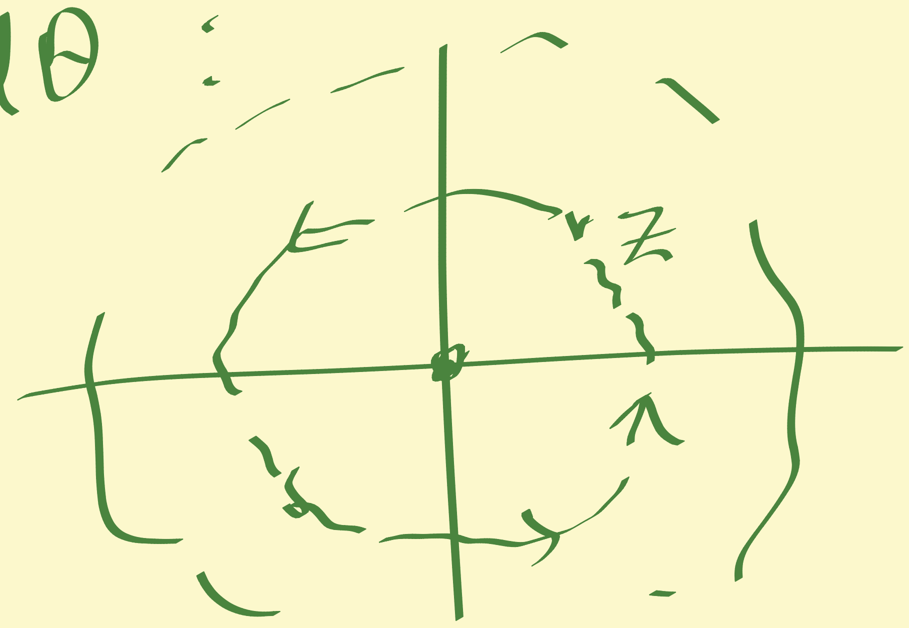
$$u(\text{center } z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\big(z_0 + \overbrace{\underbrace{r}_{\text{radius}}\cdot e^{i\theta}}^{\text{parametrize boundary}}\big)\, d\theta$$

($\exists$ formulaic version for $\mathbb{R}^n$: $\int_{S^{n-1}}$ — !?! —)

pf: ! u harmonic, then avg'd version (force rot-invt)

$$\tilde{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}\cdot z)\, d\theta :$$

center $z_0 = 0$, wolog



Claim $\tilde{u}$ still harmonic:

$$\Delta\left(\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta}\cdot z)\,d\theta\right)$$

in z

$$\stackrel{!?!}{=} \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\Delta_z\left(u(e^{i\theta}\cdot z)\right)}_{\stackrel{\text{?!}}{=}\ (\Delta u)(e^{i\theta} z)}\,d\theta$$

see Gelfand-Pettis / weak integrals

not elem.

Δ rot-invt / eqt

$$(\underbrace{\Delta u}_{=0})(e^{i\theta} z) \quad 0$$

(ach!?!)

Next: want Δ in polar coords:

$f(x\in\mathbb{R}^n) = \varphi(\underbrace{|x|}_{r})$, then

f "spherically sym." &/or "rot invt", etc.

Δ rot invt
$\Rightarrow$ again rot invt

$$(\Delta f)(x) = \left(\varphi'' + \frac{n-1}{r}\varphi'\right)(r)$$

This is Euler-type eq'n : (not const coef —)

Know const-coef homog eq'n $u'' + au' + bu = 0$

are solved by $\lambda^2 + a\lambda + b = 0$

simple λ_1 & λ_2 : $Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

(MVT $\Rightarrow$ uniqueness, etc. on intervals)

Euler-type are const-coef in different coords:

(via log & exp!: $\mathbb{R} \underset{\log}{\overset{\exp}{\rightleftarrows}} (0, +\infty)$

const coef ⟶ Euler-type

$$u'' + \frac{a}{r} u' + \frac{b}{r^2} u = 0$$

Anticipate

x^λ (not $e^{\lambda x} = (e^x)^\lambda$)

$$\lambda(\lambda - 1) + a\lambda + b = 0$$

double roots ?! : → const coef : $xe^{\lambda x}$

$\uparrow\downarrow$ exp/log

→ Euler : $\log x \cdot x^\lambda$

In 2D, eq'n $u'' + \frac{1}{r} u' = 0$

$$\lambda(\lambda-1) + \lambda = 0$$

$0 \qquad \lambda^2 = 0$: double :

$x^0 = \underline{1}$ & $\underline{\log|z|}$

$\hookrightarrow$ In 2D : most general radial harmonic is $A + B \underbrace{\log|z|}_{\text{blows up as } |z| \to 0^+}$

So $\underline{\text{const.}}$ $\Rightarrow B = 0$

on disk

$\boxed{\text{rot-inv't harm. fns on disc are } \underline{\text{consts}}}$

u harm., $\underline{\text{avg'd}}$ $\tilde{u}$, so $\tilde{u}$ = const.

rot-inv't $\underline{\text{which?}}$ $\tilde{u}(\text{center}) = u(\text{center})$

by design

$\Longrightarrow \boxed{\text{avg}} = u(0) \equiv$ mean value prop of harm. ☺

Higher dim's: $\mathbb{R}^n$

$$u(x) = \varphi(|x|) \qquad (|x| = r)$$

$$\Delta u(x) = \left(\varphi'' + \frac{n-1}{r}\varphi'\right)(r)$$

as above

$$\lambda(\lambda-1) + (n-1)\lambda = 0$$

$\lambda = 0$ is one soln & $\lambda = -(n-2)$ is other

($n=2$: get another 0 & log —)

$\Longrightarrow$ on $0 < r < +\infty$, in $\mathbb{R}^n$, gen'l radial/rotn invt/sph. sym. harm fcn is

$$\underbrace{A}_{\text{always ann. by } \Delta} + \frac{B}{|x|^{n-2}} \quad (n \neq 2)$$

~~($n=1$: $A+Bx$ ...)~~

Related to Green's functions — want $\underline{u}$ on $\mathbb{R}^n$

rot'n inv't & $\Delta u = \delta$ ("Dirac delta")

$\Downarrow$ δ really is "0 away from 0"

Example: partial info already dets almost everything

away from 0 $\quad \Delta u = 0 \quad$ $n > 2$

$\Downarrow$

know it away from 0:

$$u = {}^{\exists}A + \frac{B}{|x|^{n-2}} \qquad \left(\begin{array}{c} n=2 \\ A + B\log|x| \end{array} \right)$$

$\Delta(A) = 0$ on whole $\mathbb{R}^n$, so does not help produce δ —

under optimistic hypotheses

$n > 2$, $\quad \Delta \dfrac{1}{|x|^{n-2}} = {}^{\exists}C_n \cdot \delta$ $\quad$?!

$n = 2$, $\quad \Delta \log|x| = {}^{\exists}C_n \cdot \delta$ $\quad$?!

Mean-Value Prop. $\Longrightarrow$ Poisson's eq'n for harmonic fcn in interior, from boundary values

(Forgot: to prove MV Prop with only assp'n u ext. cont. to boundary, need to avg over slightly-smaller copies of ∂, & take limit ...)

"harm on ~~closed~~ disk"? "holo ~~on closed~~ ..."

Better: harm or holo on open ($\neq \emptyset$) & ... ext cont to boundary ?!

u harmonic on open disk, ext. cont. to closed

In 2D: use transitive action of "$U(1,1)$" Möbius on unit disk to reduce to case value @ center:

$U(1,1) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; g^{*}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

conj. transp

$\{$ holo bijection disk $\to$ itself $\}$

(Schwarz' Lemma)

(see notes for 2D)

$$u(z) = \underbrace{\frac{1}{2\pi}\int_0^{2\pi}}_{\text{artifact of avging}} \underbrace{u(e^{i\theta})}_{\partial\text{-values}} \boxed{\frac{1-|z|^2}{|z-e^{i\theta}|^2}}\, d\theta$$

!?!

thanks to Riemann

than Cauchy — :-)