$\exists$ examples/hmwk #05 2 days   26 Jan 2022

$\exists$ midterm #05 9 days

?! in-person next week: start Jan 31
Vincent 301

Harmonic fcns!

Dirichlet's Thm: Given f cont (etc. ...) on circle,
$\exists!$ harmonic u on open disk, s.t.
u ext to C^0 fcn on closed disk, & values on $\odot$ are f. "
└ given BVC or IV...

Rephrase
$$\begin{cases} \Delta u = & \text{on open disk} \\ u\big|_{\partial \text{disk}} = f \end{cases}$$

$\exists$ options!

u ext to C^0 C^1 on closed ✓ u has well-def'd precise values on ∂disk — ✓

?!

can allow disc. ∂-values — "f"

a.e. $\forall\theta$, $\lim_{r \to 1^-} u(re^{i\theta}) = f(e^{i\theta})$ etc.

proof : $\boxed{\text{rediscovers}}$ Poisson ker (Recall :

can get P-ker from MV Prop by

ch-of-vbles ~ " ~ is structural !

l.f.t. to move $0 \in \mathbb{D}$ to given $z \in \mathbb{D}$ ...

$$! \begin{pmatrix} \frac{1}{\sqrt{1-|z|^2}} & \frac{z}{\sqrt{1-|z|^2}} \\ \frac{\bar{z}}{\sqrt{1-|z|^2}} & \frac{1}{\sqrt{1-|z|^2}} \end{pmatrix} \in U(1,1)$$

= l.f.t.'s

stab disk ✓ "

Schwarz' lemma

(sends $0 \to z \in \mathbb{D}$) "

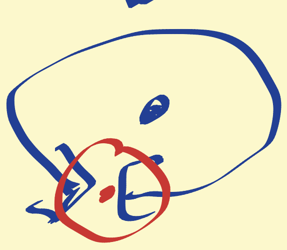
Use Fourier series ≡ universal thing :

simplest physical spaces ?

- circle — cpt, $\partial = \emptyset$
- line
- disk ?
- sphere ??

(interval $[a,b]$) $\boxed{\text{a group}}$

more elem.



!?!

not worry !

Grant "nice" fcns on circle $\mathbb{R}/2\pi\mathbb{Z} \approx [0,2\pi]$

(tbd)

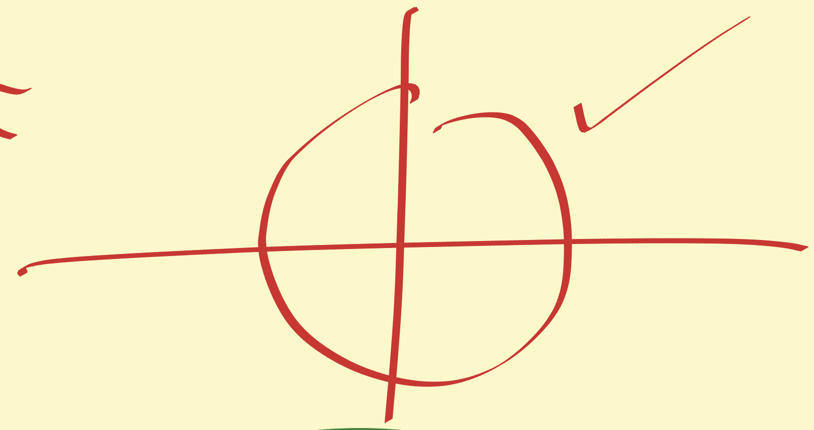
endpts identified

$\cong$

use this parametrization

$\approx \{ z \in \mathbb{C} : |z| = 1 \}$

" $\theta \to e^{i\theta}$ " $\approx$



lim !?!

$$f(\theta) = \sum_{n\in\mathbb{Z}} \underbrace{\hat{f}(n)}_{\text{F.-coefs}} \cdot e^{in\theta}$$

Great idea!

& $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-int}\, dt$

in what sense?! certainly like ptwise ✓ ☺

(?!, even for $f \in C^0$) Yikes

have "unif ptwise" for $f \in C^1$ ✓ ☺

(Crazy: deriv of $\Sigma \xrightarrow[C^0]{\times} f'$!?!)

aware!

nevermind

? f on S^1 & want harmonic u ~~w/~~ on open

$\|$ disk whose

∂(unit disk in $\mathbb{C}$)

"∂-values" (nonunique $\ast$ ---)

are given $f \in C^1$ ✓

(in on-line notes, $\exists$ intro to F.-series ~ 1828 Fourier & Dirichlet & Euler, et al —)

ch. notn to be compat with u on disk

$$f(e^{i\theta}) \overset{✓ \,"}{=} \sum_{n \in \mathbb{Z}} c_n \cdot e^{in\theta}$$

$\{$

$u|_{\partial \text{disk}}$

Observe! $e^{in\theta} = \partial\text{-value of} \begin{cases} z^n & n \geq 0 \\ \bar{z}^{|n|} & n \leq 0 \end{cases}$

reasonable (acquire technique)

Imagine?!

$$u(\overbrace{re^{i\theta}}^{z}) \overset{✓}{=} \underbrace{\sum_{n \geq 0} \hat{f}(n) z^n}_{\text{harm}} + \sum_{n<0} \hat{f}(n) \bar{z}^{|n|}$$

$\|$ $z, \bar{z}$ harm

z^n harm, $\bar{z}^{|n|}$ harm.

That is (nevermind cryptic!) harm fcn w/
"boundary values" given f. ✓ ☺

II Express this u as

discover!?

"$u(z) = \int \underbrace{P(e^{i\theta}, z)}_{\text{mystery?}} f(e^{i\theta})\, d\theta$"

boundary

?!

Poisson kernel ($\sim$ Cauchy kernel) et al

Fourier Series

⑥ Use $\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(e^{in\theta})\, d\theta$

$$u(z) \overset{\text{recop.}}{=} \sum_{n\geq 0} \hat{f}(n) z^n + \sum_{n<0} \hat{f}(n) \bar{z}^{|n|}$$

$$1/2\pi \quad = \sum_{n\geq 0} \int_0^{2\pi} e^{-in\theta} f(e^{i\theta}) z^n\, d\theta + \text{other}$$

$$\left(\sum\int = \int\sum\right) \overset{!}{=} \int_0^{2\pi} f(e^{i\theta}) \underbrace{\left(\sum_{n\geq 0} e^{-in\theta} z^n\right)}_{\text{geom series!}} d\theta + \text{other}_{\sum_{n\leq -1}}$$

$$= \cancel{(2\pi)} \int_0^{2\pi} f(e^{i\theta}) \frac{1}{1-e^{-i\theta}z}\, d\theta$$

! or

$$+ \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}\bar z}{1-e^{i\theta}\bar z}\, d\theta$$

ach!

$$\overset{!!}{=} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) \underbrace{\left(\frac{1}{1-e^{-i\theta}z} + \frac{e^{i\theta}\bar z}{1-e^{-i\theta}\bar z}\right)}_{\text{? Poisson ker ?}} d\theta$$

Simplify:

$$\frac{1 - \cancel{e^{-i\theta}\bar z} + \cancel{e^{-i\theta}\bar z} - e^{-2i\theta} z\bar z}{(1-e^{-i\theta}z)(1-e^{i\theta}\bar z)}$$

:)

$\exists$ choices

$$= \frac{1-|z|^2}{\underbrace{|z-e^{i\theta}|^2}}$$

$$x \cdot \bar x = |\cdot|^2$$

"just alg + geom series" $P(e^{i\theta}, z)$ ✓ :)

NB: uniqueness ?!? If $u \equiv 0$ on bdy,

then $u(\text{center}) = 0$ ✓ ----

1^{st} pf of Poisson fml $\Longrightarrow$ $u = 0$ throughout

(Max princ: harmonic u has $|u|$ max on ∂ .... & if attained in interior, $u =$ const

~ hdw ✓ :))

Cor: our heuristic for discovering Poisson ker is entirely correct: no ambiguity ✓ :)

(there can be at most one such)

.. & we made one ..

so that's it ✓ :)

$\boxed{\textbf{Thm}}$ on arb. (conn., $\neq \emptyset$) open $\mathcal{U} \subseteq \mathbb{C}$

w/ not-too-crazy™ boundary, "barriers"

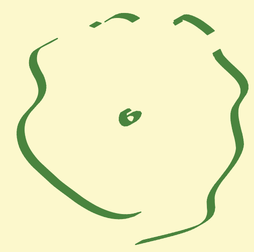
given C^0 f on ∂, $\exists!$ harmonic u ?!

in interior ext cont. to ∂ w/ values f.

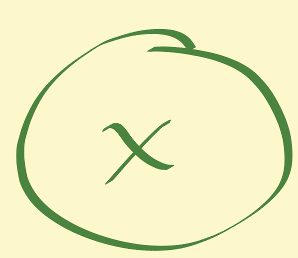
NB: $\boxed{\text{R. map thm}}$ ($\exists$ holo isom. ϕ

simply-conn $\mathcal{U} \subseteq \mathbb{C}$

$\mathcal{U} \neq \mathbb{C}$ to open disk)

reduces some cases to  ✓

Next time: u harm on annulus

(e.g., )