Class on-line for 2 more weeks 28 Jan 2022

!?! "

Write-ups today ~ or ... "soon" :)

$\exists$ exam 05 Fri. Feb 04 $\equiv$ remote + honor system

(no "computation-intense")

Harmonic fcns :

Recall (+ notes) on a disk, classify them!

$\Longleftrightarrow$ Dirichlet problem :

all h-fcns on $\{|z|<1\}$ are expressible

(uniquely!) as

$$c_0 + \sum_{n=1}^{\infty} \left(a_n z^n + b_n \bar{z}^n \right)$$:)

!!

for nice behavior @ boundary,

the $\lim_{r \to 1^-}$ is F.-series $c_0 + \sum_{n=1}^{\infty} \left(a_n e^{in\theta} + b_n e^{-in\theta} \right)$

Recall: F.-series big $\Rightarrow$ existence

MV Prop + moving pts $\Rightarrow$ uniqueness ✓

by $u(1,1)$

(?) Distributions (= "generalized fcns")

on circle do have F.-expns (cvgt in Dirichlet-but-legit way?)

... same arg. gives harmonic

u on open disk with $\lim_{r \to 1^-}$ = boundary distr.?

(in $\boxed{\text{top}}$ of ... ?! ... not precise ...)

Robustness —

Compare: F.-series !?! ∂-values of holo disk?

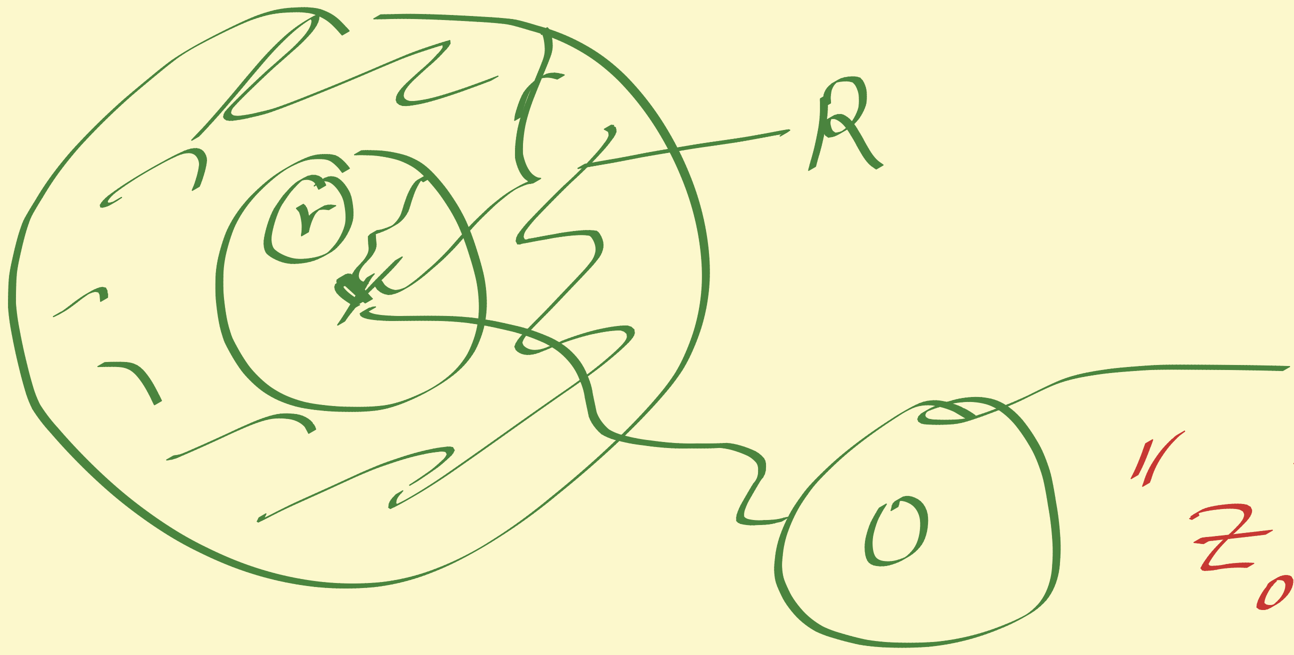
$u(z)$ = one-sided

$$\sum_{n=0}^{\infty} c_n z^n \xrightarrow[|z| \to 1^-]{} \sum_{n=0}^{\infty} c_n e^{in\theta}$$

$\neq$ everything

no $n<0$ terms ?!?

On annuli



"z_0" $= 0$

Holo: $\exists(!)$ Laurent exp'n

$$\underbrace{\sum_{n\in\mathbb{Z}}}_{\text{2-sided}} c_n z^n \quad \text{as } |z| \to R^+,$$

?!? $\longrightarrow$ $\underbrace{\sum_{n\in\mathbb{Z}} \left(c_n \cdot R^n\right) e^{2in\theta}}$ 2-sided F.-series ✓ ☺

versus 1-sided F.-series for P.S.

($\Longrightarrow$ up to cvge....) any "nice" fcn on $|z| = R$ is achievable as ∂-value of holo fcn on ann.!

(! option of allowing trouble in inner disk gives lots more ~~fcn~~ holo fns!)

& $\lim\limits_{|z| \to r^+} \ldots \overset{\text{sim.}}{=} \sum\limits_{n \in \mathbb{Z}} (c_n \cdot r^n) \cdot e^{in\theta}$

2 sided

holo = any fcn w/ ... F.-series - (cvge ... ✓)

But/and both inner & outer circle-values simult.? — no

But w/ harmonic fcns:

claim at least $\exists$ harmonic ✓

✓ $c_0 + c_1 \log|z| + \sum\limits_{n \in \mathbb{Z}} \left(a_n z^n + b_n \bar{z}^n \right)$!

$n \neq 0$

$z^0 = \bar{z}^0$

Rot-inv't

(Ex: a "nice" sum of harmonic things is harmonic;

~ analogous pf for holo via Cauchy fml;

Poisson fml (MVP rep))

Such series @ $|z| = R$ & $|z| = r$ are

$$\underbrace{c_0 + c_1 \log R}_{\text{const}} + \underbrace{\sum_{n \in \mathbb{Z}} \left((a_n R^n) e^{in\theta} + (b_n R^n) e^{-in\theta} \right)}$$

$$\| $$

$$\sum_{n \in \mathbb{Z}} e^{in\theta} \cdot \left(\underbrace{a_n R^n + b_{-n} R^{-n}}_{\text{coef}} \right)$$

& sim. @ $|z| = r$

(in typeset notes !?!)

Dirichlet problem for  : given "nice" fcn on both inner & outer ○'s, $\exists!$ harmonic fcn on interior w/ those ∂-values

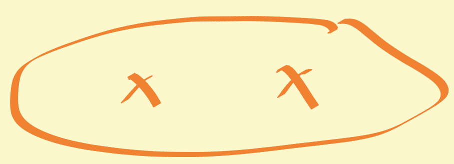
(one way for $\exists$: imitate F.-series arg for disk. —)

∃ **Thm**: for open $\neq \emptyset$ with "nice" (e.g., finitely-piecewise C^1) boundary, given C^0 on ∂, $\exists!$ harmonic fcn on interior of those ∂-values.

no symmetry !!

∃ explicit solns/resolutions for □'s

▭'s

& ⊙'s (not ?!,  ?!)

(not ?! △'s)

Uniqueness: show harmonic u on $\mathbb{C} - \{0\}$ w/ $u(\mu \cdot z) = \mu^{n} \cdot u(z) \quad \forall |\mu| = 1$

rot sym/eqce (not invce)

(0^{th}) is of the form $\boxed{u(re^{i\theta}) = e^{in\theta} \cdot f(r)}$;

$\mathbb{R}^{2}$ $\qquad r > 0 \qquad$ $n-1$ in $\mathbb{R}^n$

& (1^{st}) $\quad f'' + \frac{1}{r} f' - \underbrace{\frac{n^2}{r^2}}\, f = 0$

$(n = 0$, easy$)$

Implications: Euler-type, indicial eqn

$$\lambda(\cancel{\lambda - 1}) + \cancel{\lambda} - n^2 = 0$$

$$\lambda = \pm n$$

$\surd$ $\circlearrowleft$

$e^{in\theta} r^{n}$ & $\underbrace{e^{in\theta} r^{-n}}$ are ~~the~~ 2 l.i. solns

$z^n \qquad \bar{z}^{-n}$

$(n \neq 0)$

$(n = 0$: consts & log$)$

uniqueness via MVT calc.

Easy pf that arb. fcn u w/ $u(\mu\cdot z) = \mu^n \cdot u(z)$

$\forall \mu : |\mu| = 1$ is of the

form $u(re^{i\theta}) = \underbrace{e^{in\theta}}_{(e^{i\theta})^n} \cdot u(r)$:.

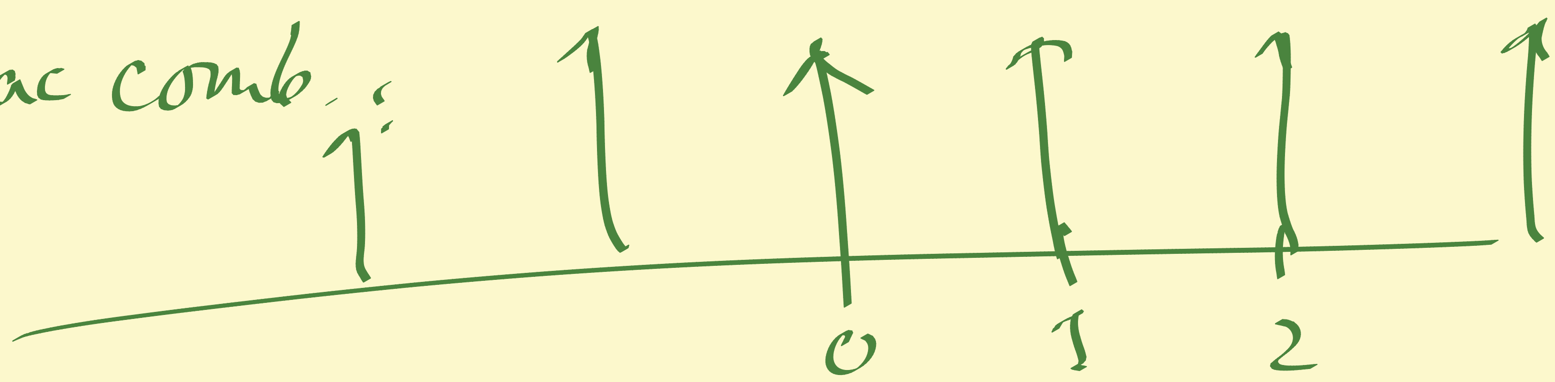
"f" above

$$\text{Right: } u(\underbrace{e^{i\theta}}_{\mu} \cdot \underbrace{\overset{(0,+\infty)\,\ni}{r}}_{z}) \underset{\text{hyp}}{=} (e^{i\theta})^n \cdot u(r) \checkmark$$

z

Bonus: periodic δ at $1 = e^{i\cdot 0}$ on circle

fun Dirac comb:



has F. expn $\boxed{\sum_{n\in\mathbb{Z}} 1\cdot e^{in\theta} = \delta_{\mathbb{Z}}}$

not cvge ptwise ✓ but does cvge ---ion

Can we make harmonic fcn u on $|z|<1$

w/ $\delta_{\mathbb{Z}}$ as ∂-values !?! :

$$\sum_{n\in\mathbb{Z}} a_n e^{in\theta} \xrightarrow[\text{Dir. }]{} \underbrace{a_0 + \sum_{n=1}^{\infty}\left(a_n z^n + a_{-n}\bar{z}^n\right)}$$

∂-values — hen.

w/ all a_n's $= 1$: $1 + \dfrac{z}{1-z} + \dfrac{\bar z}{1-\bar z}$ geom series !

$$\overset{\text{alg.}}{\underset{!?}{=}} \frac{1-|z|^2}{|1-z|^2}$$

$\boxed{\text{on-line next wk}}$

$\delta_{\mathbb{Z}}$ shifted by θ ----

$$\frac{1-|z|^2}{|e^{i\theta}-z|^2} \overset{!?!}{=} \text{Poisson kernel !?!}$$