Practice/example write-up soon 31 Jan 2022

Yes $\exists$ on-line exam 05 this Fri:

I'll email you PDF —

* keep file sizes down —

Yes, class on-line/remote this & next week. —

Harmonic fcns .... $\longleftrightarrow$ holo fns —

& on $\mathbb{R}^n$ .... & on "Riemannian manifolds"

$\Delta u = 0$

(& heat $\Delta u - \frac{\partial u}{\partial t} = 0$

wave $\Delta u - \frac{\partial^2 u}{\partial t^2} = 0$

Helmholtz $\Delta u - \lambda u = 0$

– – – – –)

! Ring of const-coef translation-inv't, rot'n-inv't diff ops on $\mathbb{R}^n$ = {polys(Δ)} ! "

1840

"Liouville Thm for harmonic": bdd "$\Delta u = 0$"

$\mathbb{R}$-valued

or

$\mathbb{C}$-valued

(C^2-) harmonic fcn on $\mathbb{R}^n$ is const.

(~ bdd entire fcn on $\mathbb{C} \simeq \mathbb{R}^2$ is const.)

Easy: $u \in C^2$

then Δu can be eval'd "classically": derivs are just those definitional limits of differences

Thm: In fact, because Δ is "elliptic diff op"

$\Rightarrow \Delta u = 0$ for dist. u

$\Downarrow$

$u \in C^\infty$ ✓

$$\frac{\partial}{\partial x_j} f(x) = \lim_{h \to 0} \left(\frac{f(x_1, \ldots, x_j + h, \ldots x_n) - f(x_1, \ldots x_n)}{h} \right)$$

"classical" defn (dist. from "distributional") pf: Use Fourier transform $\mathcal{F}$, on $\mathbb{R}^n$

$$\hat{f} = \mathcal{F}f(x \in \mathbb{R}^n) = \int_{\mathbb{R}^n} e^{-2\pi i \langle x, y \rangle} f(y)\, dy$$

i.p. on $\mathbb{R}^n$

interchanges diff ops & mult'n-by-polys:

smooth

$$\Delta u = 0 \Longrightarrow \boxed{(-2\pi i)^2 \cdot |x|^2 \cdot \hat{u} = 0}$$

$= \hat{0}$

?!?

Convergence?

extend $\mathcal{F}$ on * 2024

~ L. Schwartz ~1950

A. Grothendieck ~ 1950's

$\Downarrow$

$\operatorname{spt} \hat{u} = \{0\}$

not just "bdd C^2"

but $\boxed{\text{"tempered distribution"}}$

Taylor-series Thm

δ & its derivs !?!

$\hat{u}$ Thm $=$ polys (!!!)

So, necessarily wh bdd $\Delta u = 0 \Longrightarrow u = $ poly: harmonic poly

So, if ptwise bdd, it's const. ✓ ☺

§§ Analogous pf for Liouville thm holo bdd on $\mathbb{C}$:

u s.t. $\bar{\partial} u = 0$ ($\equiv$ Cauchy-Riemann)

Classical: u is C^1? (Cauchy-Goursat works w/o assumption of cont. derivs! ☺)

u tempered distn: $\bar{\partial} u = 0$

$\Downarrow$ F.T.

$\begin{cases} \bar{\partial} \text{ ms } \times z \\ \partial \text{ ms } \times \bar{z} \end{cases}$

∂u_i's

$z \cdot \hat{u} = 0$

$\Downarrow$? thm

$\operatorname{spt} \hat{u} = \{0\}$

$\Downarrow$

$\hat{u} = \delta$ or deriv's

$u = \text{poly}(x, y)$: bdd $\Longrightarrow$ const.

Re(z) Im(z)

✓ !!!

! F.T. + distn $\Longrightarrow$ stronger conclusion about holo than Cauchy–Goursat !

(Illustrates efficacy of "distn-theory":

See Heaviside ~ 1900

Dirac ~ 1930's !

& current physics ✓)

Ex: (see hmwk) of utility of ∂ & $\bar\partial$ ops:

harmonic $\circ$ holo = harmonic

(while $\boxed{\text{holo} \circ \text{harm} = \text{not nec. harmonic}}$,

& ! ∄ harm. $\circ$ harm $\neq$ harm,

tho' holo $\circ$ holo = holo)

If correct, $\boxed{\exists \text{ easy examples}}$: awareness! :-)

"holo" $\neq$ id ✓ silly

$z \to z^2$?! ; & harm $u(x,y) = x$

$z \to (\mathrm{Re}(z))^2 = x^2 \neq$ harmonic:

$\Delta(x^2) = 2 \neq 0$ ✓ :-)

Otoh, use $\partial, \bar\partial$ (Δ basic props —!?!)

$$\Delta = 4\partial\bar\partial = 4\bar\partial\partial$$

C.-R.

h harm. & f holo $\left(: \ \partial\bar\partial h = 0, \ \underbrace{\bar\partial f = 0}\right)$

$$\Delta(h\circ f) \overset{\checkmark}{=} \partial\left(\bar\partial(h\circ f)\right)$$

chain rule

???

$$\overset{!?!}{\underset{\text{ok}}{=}} \partial\left((\bar\partial h\circ f)\cdot\bar\partial f\right)$$

$\underbrace{\qquad\qquad\qquad}_{0}$

caution :-)

?!? :-)

yes :-)

prank (see discussion)

appears to show $h\circ f$ is holo ?!

Feasible, less comp-intensive:

Recommend: prod. expn of $\cos\pi z$?!?

not just translate that of $\sin\pi z$ ✓

ultra-classic

+ partial fraction / 1st Mittag-Loeffler thm

(in $\mathbb{C}^1$, as opposed to $\mathbb{C}^n$ $n > 1$, can be arb.!)

of

$\tan \pi z$, $\cot \pi z$, ... $\sec^2 \pi z$, ...

(~ method as in notes)

→ natural guess is ~~provably~~ correct:

trig thing − natural partial frac. guess = entire & $\downarrow 0$ as $\operatorname{Im}(z) \to \pm\infty$

$\Downarrow$ Liouville

$\equiv 0$ ✓ ☺

not easy for Euler in 1735 —

Basel —

Not'n for proving rot-invce of Δ:
terminology

vs. equivariance

let R = a rot'n in $\mathbb{R}^n$ $\equiv$ R is in $SO(n,\mathbb{R})$

$\left\{ \begin{array}{l} \text{linear maps} \\ R: \mathbb{R}^n \to \mathbb{R}^n \text{ s.t.} \end{array} \right.$

allows reflection & orientation-reversing

$R^T R = 1_n$ &

$\det R = 1$ }

elem. reflections.

$$\boxed{\Delta(f \circ R) = (\Delta f) \circ R}$$

(not ~~$\Delta(f \circ R) = \Delta f$~~, nor ~~$\Delta f = \Delta f \circ R$~~

only possible for f Rot-inv't

"

(not about $4\partial\bar\partial = \Delta$ ...)

$\exists$ economical not'n?! (see discussion ...)

AWARENESS

Say Δ & Rot commute

$$\Delta(\underline{\text{Rot}}\ \underset{\text{dummy}}{f}) = \underline{\text{Rot}}(\Delta f) \quad \text{not'n}$$

$$\Delta \circ \text{Rot} = \text{Rot} \circ \Delta \quad ! \checkmark \ \text{"commute"}$$

Let linear endos of $\mathbb{R}^n$ act on diff ops by R

$$R(\Delta) = \underbrace{R \circ \Delta \circ R^{-1}}_{} \quad (!)$$

"equivariant" becomes "invariant" ! ✓

$\boxed{\text{Fri exam on-line}}$ ✓