$\exists$ hmwk ex 06 "due" Feb 25 07 Feb 2022

& midterm 06 Mar 4 ($\leftarrow$ Fri before Spr. break)

More classic cx an. esp. relevant to/motivated by

zeta function (& related!)

(filling in all crazy gaps in R.'s 1858 paper ...)

missing

Riemann Hypothesis —

all 0's of $\zeta(s)$ in $\underbrace{0 \leq Re(s) \leq 1}$

"the crit. strip"

sense?!

are on $\underbrace{Re(s) = 1/2}$

implications?!

"the crit. line"

subtle ...

a big deal ...

✓ Harmonic fcns ... will reappear! "Poisson-Jensen fml"

wt'd count of

0's of holo fcn (via

$(f \neq 0;)$ (log Re f) harm

Mean value prop! ✓"

Poisson fml

Products (see notes #12):

Weierstraß products
& Thm! Hadamard products — for ζ
1893

(compare $\boxed{\text{Mittag-Loeffler}}$ ("partial frac")
& arb. polar parts @ discrete
& closed set of
pts)

!?!

Weaker requirement could "poles @ $z_1, z_2, \ldots$"
exactly

$$\sum_n \left(\frac{c_n}{z - z_n} + \text{holo} \right)$$

where c_n suff. rapidly decr"

?!?

(ex: show discrete closed subset of $\mathbb{R}^n$ is countable)

sequence

But w/ spec'd res ?!?
↑↑ tcs ?!

Sim. issue w/ "Weierstraß product thm":

Thm: given $z_1, z_2, \ldots$ discrete & closed

can appear with multiplicies ✓ "

Can make "prod expn" to create entire fcn with (exactly) those 0's

(can write the most general such:

given 2 such entire fcn f, g,

f/g has removable sing.,

so is* entire, w/o 0's, so

is $e^{h(z)}$ for some entire h.

And h will create another entire — from given one by

$$F(z) = e^{h(z)} \cdot f(z)$$ ✓)

Explicitly (part of content?):

Polys f w/ 0's $z_1, \dots, z_N$ (w/ mults)

algebra

$$f(z) = \underbrace{\exists c}_{\neq 0} \cdot \prod_{n=1}^{N} (z - z_n) \checkmark$$

☺

Heuristic: for ∞-many z_n's, ~~never~~ seldom converge ☹

e.g., @ $z = 0$??? ☹

$\prod z_n$? $\left(z_n = 1 + \frac{1}{n^2} \longrightarrow 0 \right)$ acc. pt.

?!?

$z_n = n$?

impossible $\Longrightarrow$ modify algebra $\rightarrow$

Try $\quad z^m \prod_n \left(1 - \frac{z}{z_n}\right) \stackrel{?}{=} f(z)$

for m-fold 0 of f $\qquad z_n \neq 0$

E.g., $z_n = n^2$, nice cvgce :)

Sometimes can group: $\quad \underbrace{\prod \left(1 - \frac{z^2}{n^2}\right)}_{\pm n}$

for $\frac{\sin \pi z}{\pi z}$

General ?!? Need $\boxed{\text{"cvgce factors" } \Phi_n}$

hols, do not vanish at all, &

$z^m \prod_n \left(1 - \frac{z}{z_n}\right) \cdot \Phi_n(z)$ does cvge

Want $\Phi_n(z) \simeq \left(1 - \frac{z}{z_n}\right)^{-1}$?!?

but no 0's

?! $\Phi_n(z) \approx e^{\underbrace{-\log\left(1-\frac{z}{z_n}\right)}_{\text{not entire}}}$!

Refine: $\Phi_n(z) = e^{p_{N_n}(z/z_n)}$

$p_N(z) = N^{th}$ Taylor poly of

$-\log(1-z)$ @ $z=0$

& Thm: given z_n's, $\exists\, N_n$'s st.

$$z^m \prod_n \left(1 - \frac{z}{z_n}\right) \underbrace{e^{p_{N_n}\left(\frac{z}{z_n}\right)}}_{\text{never } 0}$$

poly entire

cvsces well !

If cvges, has desired 0's

Works ?! :)

See notes for pf ✓ ☺

Most general such f is

$$\underbrace{e^{\underline{g(z)}}}_{\text{non-0}} \cdot z^m \cdot \prod_n \left(1 - \frac{z}{z_n}\right) \cdot e^{-P_{N_n}\left(\frac{z}{z_n}\right)}$$

?!

entire

ambiguous

+ increasing N_n's, only helps conv ✓

(see pf)

⟶ necessitates altering $g(z)$

"genus" of such expn is $\max\left(\deg \text{poly } g, \sup_n N_n\right)$ ☹

or is $+\infty$ for g non-poly

&/or for $\sup N_n = +\infty$

(genus of "expression")

genus of a fcn = inf of genus of all expn for f

Weierstraß

temp

If f has genus g, then f has expn

$$f(z) = e^{\text{poly degree} \le g} \times z^{m} \times \prod_{n} \left(1 - \frac{z}{z_n}\right) e^{P_g\left(\frac{z}{z_n}\right)}$$

$\exists$

same

uniform

"λ" in notes ??!??

"order" of entire fcn f is smallest $h \in \mathbb{R}$ st

$$\forall \varepsilon > 0, \quad |f(z)| \le e^{|z|^{h+\varepsilon}} \text{ for large } |z|$$

no const.

Alt.

$$\forall \varepsilon > 0, \exists C_\varepsilon \text{ st } |f(z)| \le C_\varepsilon \cdot e^{|z|^{h+\varepsilon}}$$

Alt.

$$\forall \varepsilon > 0, \qquad |f(z)| \ll_\varepsilon e^{|z|^{h+\varepsilon}}$$

"\ll"

context ???

(in notes "genus" is h, "g" was reserved for fcn in exp — ??)

Thm genus ≤ order < genus + 1 ✓ ☺
(of growth)

The growth rate/"order" of f nearly dets its "genus" in W-prod sense! ☺

∃ pot. to clarify prod expn of ζ (in terms of its 0's ! (not Euler-)

Lots need growth & $(s-1)\zeta(s)$ ✓ mero cont'ine ; entire