Cor of pf of Weierstraß Π

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! ($\exists$ revision of notes 12):

for f fcn w/ 0's z_n, of genus h, then

$$\boxed{\sum_n \frac{1}{|z_n|^{h+1}} < \infty}$$ ✓ :-)

$z_n \neq 0$

In pf. was $\sum_j \frac{1}{|z_j|^{m_j+1}} < \infty$ $\exists$

$\exists$ genus $< \infty$

$\Updownarrow$

$\exists$ finite sup to m_j's ✓

Conversely (used to prove W's thm)

also

$$\prod_j \left(1 - \frac{z}{z_j}\right) \cdot e^{P_{m_j}(z/z_j)}$$

nicely cvgt ✓

($\Rightarrow$ W's)

Preview of applics to $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $Re(s) > 1$

will see $\underbrace{\xi(s)}_{\text{"completed } \zeta\text{"}} = \underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\text{"the gamma factor"}} \times \zeta(s)$

\xi

~~$\nearrow s$~~ ; no. th. "

$\underbrace{s(1-s)}_{\text{ann. poles}} \cdot \xi(s)$ is entire & inv't under $s \longleftrightarrow 1-s$

zero

has an. cont'n to — here (not about paths —)

later: Monodromy, "slitting" plane

not now?

$\exists$ entire fcn $F(s)$ st.

$$\boxed{F\Big|_{Re(s)>1} = s(1-s)\,\xi(s)}$$

= "all at once"

Think of $1+s+s^2+\ldots$ at first $|s|<1$

an. cont by rewrite to $\frac{1}{1-s}$ to $\mathbb{C} - \{1\}$

Don't know "order of growth" λ $\big(h \leq \lambda < h+1\big)$ — h: genus, λ: order-of-growth

yet: needed work? why want...

Still, W' thm:

Saw Γ forces ~~$\zeta(0)$~~ $\zeta(-2) = \zeta(-4) = \dots = 0$ — "trivial 0's"

$$\underbrace{s(1-s)}_{\text{ann. } \infty' \text{s}}\ \underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\Gamma\text{-factor}}\cdot \zeta(s) =$$

$$= e^{\text{entire}(s)} \cdot \prod_{\rho_j} \Big(1 - \frac{s}{\rho_j}\Big) e^{-P_{n_j}(s/\rho_j)}$$

$\exists\, n_j$'s

ζ etc. $\neq 0$ at $s = 0$: no "s^m"

contrad. not'n! "

(not s_j ??)

"non-trivial 0's of ζ"

cancel ∞'s of $\Gamma(s/2)$ so that $\zeta(s)$ has no 0's @ ~~0~~ $-2, -4, \dots$

!

That is, have an inf. prod for ~~ξ~~ ξ ~~...~~
in terms of its 0's ✓ :)

(Riemann imagined —)

Riemann (no pf — until 1908 : Lindelöf. — .
1858–9 Phragmen–Lindelöf, etc.)

(anachronistic term.) $\xi \times s(1-s)$ (= entire)

(in effect) $\xi \times s(1-s)$ is genus 1 !?!

(True — 1908)

$$s(1-s)\,\xi(s) = e^{\overset{\exists}{a}s+\overset{\exists}{b}} \times \cancel{s^n} \times \prod_\rho \left(1-\frac{s}{\rho}\right)\underbrace{e^{s/\rho}}_{\text{convergence factor}}$$

convergence factor ✓ :)

+ fcnl eqn $\xi(1-s) = \xi(s)$

$\Longrightarrow$ if ρ is a 0, then $1-\rho$ is a 0 of ξ

So group

$$s(1-s)\xi(s) = e^{as+b} \times \prod_{\substack{\rho \\ \text{mod} \\ \rho \longleftrightarrow 1-\rho}} \left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{1-\rho}\right) \underbrace{e^{z/\rho} e^{z/1-\rho}}_{?!}$$

$$\frac{z}{\rho} + \frac{z}{1-\rho} \overset{\text{alg!}}{=} z \cdot \frac{1}{\rho(1-\rho)}$$

Cor of W's Thm, $\sum \frac{1}{|\rho|^{2}} < +\infty$ $\quad 2 = \underbrace{1}_{\text{genus}} + 1$

$$\Longrightarrow \sum \frac{1}{\rho(1-\rho)} = \underline{\text{abs. cvgt}} \;!!$$

So can(!) collect all $e^{z/\rho} \cdot e^{z/1-\rho}$ factors

on their own!

$$s(1-s)\xi(s) = e^{as+b} \cdot \underbrace{e^{s \boxed{\sum^*_{\rho} \left(\frac{1}{\rho(1-\rho)}\right)}}}_{\substack{\| \\ \text{a number}}}$$

$e^{(a')s}$

So

$$s(1-s)\xi(s) = e^{\overset{\text{other}}{as+b}} \prod_{\rho \bmod \rho \leftrightarrow 1-\rho} \left(1-\frac{z}{\rho}\right)\left(1-\frac{z}{1-\rho}\right)$$

Cvgce factors "gone" ✓ :)

means grouping

?!?!

$$\text{“} e^{as+b} \cdot \prod_{\rho}^{\#} \left(1 - s/\rho\right) \text{”}$$

Then (!)

Riemann: We have 2 $\prod$-expns for ζ/ξ:

Euler (in $\mathrm{Re}(s)>1$) & Hadamard (on $\mathbb{C}$)

primes

ρ's

$$s(1-s)\pi^{-s/2}\Gamma(s/2)\prod_{p}\left(\frac{1}{1-p^{-s}}\right) \overset{!!}{=} e^{as+b}\prod_{\rho}^{\#}\left(1-\frac{z}{\rho}\right)$$

prime

Extract info ?! ???

another Euler factor! ~ completion $\mathbb{R}$ of $\mathbb{Q}$

~ completion $\mathbb{Q}_p$ of $\mathbb{Q}$

$\exists$ factor for each completion $\mathbb{Q}$:)

Until Hardy-Littlewood ~1910±, not known whether $\exists\ \infty$-many (non-trivial) 0's of ζ

$= 0$'s of ξ

∞'s of Γ-factor cancel trivial 0's of ζ ✓ ☺

"How could there not be ?!?" ☺

→ H.-L. $\exists\ \infty$-many 0's of ξ !

Use (Hadamard, Phragmen-Lindelöf 1908)

$s(1-s)\xi(s)$ is ~~genus~~ order 1 & H's genus $\leq$ order $< h+1$
h λ

stronger than "genus $= 1$"

$+\ \underbrace{\xi(s)}_{s(1-s)} = \underbrace{\xi(1-s)}_{s(1-s)}$

$\Longrightarrow$ $s(1-s)\,\xi(s)$ is inv't under $s \longleftrightarrow 1-s$

$F(z) = (z+\frac{1}{2})(\frac{1}{2}-z)\,\xi(\frac{1}{2}+z)$ is inv't under $z \longrightarrow -z$;

$\underbrace{\xi}_{\Xi\text{-fcn}}$

it's even

So $F(\sqrt{z})$ = entire, order $= 1/2$ $\Big\}$ — $\otimes$

power series! ☺

$\Longrightarrow$ $\underset{\mathbb{Z}_{\geq 0}}{\overset{F(\sqrt{z})}{\text{genus}}} \leq \frac{1}{2} < \overset{F(\sqrt{z})}{\text{genus}} + 1$

$\Longrightarrow$ $\boxed{\text{genus} = 0}$ $\boxed{\text{for } F}$

$$F(\sqrt{z}) = e^{a} \cdot \underbrace{\prod_{z_j}\left(1 - \frac{z}{z_j}\right)}_{\text{unif conv't}}$$

unif conv't. ✓

don't need conv'ce factor ✓ ☺

! if $\exists < \infty$ z_j's, then $F(\sqrt{z}) = \text{poly} \Longrightarrow F(z) = \text{poly}$ $\longrightarrow$

but $\underbrace{\pi^{-s/2}}_{\substack{s(1-s) \\ \text{poly} \\ \sim s^2 \\ \text{exp decay}}} \underbrace{\Gamma(s/2)}_{\sim\ s^{s-\frac{1}{2}} \gg \text{any fixed exponential}} \cdot \underbrace{\zeta(s)}_{\downarrow\ 1\ \checkmark}$ as $s \in \mathbb{R}$, $s \to +\infty$

Stirling's!

(see "asymptotic"!)

whole $=$ super-poly growth -- so

can't be a poly

$\Downarrow$

$\exists\ \infty$-many non-trivial 0's of $\zeta(s)$ ✓ ☺

"doing complex analysis"