Saw H.'s $\prod$-thm : 11 Feb 2022

genus $\leq$ order of growth $<$ h+1
h λ

$(h \in \mathbb{Z})$
$\vee$
0

can rewrite to get rid of Γ - - - - DON'T*

$+$ $\boxed{s(1-s)\,\xi(s) \overset{!}{=} \text{invt under } s \longleftrightarrow 1-s}$

& growth-order 1 (iow)

ext'd $\zeta(s)$
$\|$
$\underbrace{\pi^{-s/2}\,\Gamma(s/2)}_{\Gamma\text{-factor}} \cdot \zeta(s)$

$\exists$ ∞-many non-trivial 0's of $\zeta(s)$! :)

(iow significance !?!)

Preview

grouping

Hadamard (1858-9) Euler

$$e^{as+b} \prod_{\rho} \left(1-\frac{s}{\rho}\right) \;=\; s(1-s)\,\pi^{-s/2}\,\Gamma(s) \prod_{p} \frac{1}{1-p^{-s}} \;\; ??$$

cvgt $\forall s \in \mathbb{C}$!!??!! "∞" Re(s) > 1

$\xi(1-s) = \xi(s)$: ($\underset{\approx}{\&}$ $\underbrace{s(1-s) \times \text{it} = \underline{\text{entire}}}$)

"fcnl eq'n"

(another) pf an. cont'n

≷ not via paths & homotopy

"☺"

contrast

F is an an. cont'n of f (= holo)
when dom F ⊇ dom f

& $F|_{\text{dom } f} = f$

Today: an. cont'n via "rewriting"

NB In general, holo fcns do not have an. cont'ns to larger domains !!!??
(different in $\mathbb{C}^2, \mathbb{C}^3, \ldots$)

≷ usefully — no magical general method

NO General method / criterion!

~ tho', e.g., conj'd that "all L-fcns" have an. contins ~ ~ afc ~

?!?! :-)

Even simple things may fail to have an. contin: Estermann phenom (see earlier notes: complete those pfs need some of current stuff about 0's of ζ!)

1928

Kurokawa 1980's

$$\boxed{\sum_{n=1}^{\infty} \frac{\text{sum of squares of divisors}(n)}{n^s}}$$

has "natural boundary" along $Re(s)=0$

Crazy

sum of divisors
or
no. of divisors

} have an. contin !?!

An. Contin = serious (& useful ~ $\zeta(s)$ ion)

Yes, already Riemann $\Gamma(s)\zeta(s) = \int_0^\infty y^s \frac{1}{e^y - 1} \frac{dy}{y}$ (?!)

Trick (see notes earlier) convert to Hankel/keyhole contour

Bernoulli

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§ More provocative an. contn using a "theta fcn"

(notes out-of-order ; see "Riemann & ζ" 12f later section)

let (?!)

$$\Theta(y) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 y} = \text{"a theta series"}$$

variation on std

$n=0$?!?

integral representation

Riemann's $\int$-presentation

(from where? Abel, Jacobi Fourier, et al studied & Euler ?!)

Riemann was aware ...

$$\underbrace{\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s)}\ \overset{\text{first}}{=\!=}\ \int_0^\infty y^{s/2} \cdot \frac{\Theta(y) - 1}{2} \frac{dy}{y}$$

$Re(s) > 1$

?!?

$\pm n \to$ same

second prove/recall? (ion)

funl eq'n of $\theta(y)$: $\boxed{\theta(1/y) = y^{\pm 1/2}\,\theta(y)}$ pf will discover

Poisson Σ'n (nice f)

$$\boxed{\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)}$$

= Riemann-Roch !!!

third

Use funl eq'n of θ to get an. contn & funl eq'n of ζ / ξ

Fourier series ! : ptwise f periodic (nice)

$$f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\cdot e^{2\pi i n x}$$

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x)\,dx$$

Fourier-Dirichlet 1820±

Trick? $\displaystyle\int_0^\infty y^{s/2}\,\frac{\theta(y)-1}{2}\,\frac{dy}{y} = \int_0^1 + \int_1^\infty$ ✓

nat. mult-invt measure!

Claim: $\displaystyle\int_1^\infty$ is entire in s;

(will) use fcnl eqn θ to convert $\int_0^1$ to an $\underbrace{\int_1^\infty}_{\text{entire}} + \underbrace{\text{elem}}_{\text{poles}}$

pf:

it $\displaystyle= \int_1^\infty \underbrace{y^{s/2}}_{\text{poly growth}} \underbrace{\sum_{n=1}^\infty e^{-\pi n^2 y}}_{\text{show is exp. decay @}\infty} \frac{dy}{y}$

+ usual Cauchy-th. Lemmas

ex! show $|\Sigma| \le \exists C\cdot e^{-y}$ ___ ☺

step

Convert $\int_0^1$ to $\underbrace{\int_1^\infty}_{\text{entire}}$ + 2 elem terms

poles: explicit ✓!

side-channel info!!

$\Downarrow$

ch. vbles y by $1/y$, see what happens ✓!

ha!

$$\int_0^1 y^{s/2}\,\frac{\theta(y)-1}{2}\,\frac{dy}{y} \underset{\substack{y \text{ by} \\ 1/y}}{=} \int_1^\infty y^{\frac{-s}{2}}\,\frac{\theta(1/y)-1}{2}\,\frac{dy}{y}$$

also invt under $y \to 1/y$ ✓!

we'll find out!

$$= \int_1^\infty y^{\frac{1-s}{2}}\left(\frac{\frac{1}{\sqrt{y}}\theta(y)-1}{2}\right)\frac{dy}{y}$$

rearrange to look like orig $\int_1^\infty$

$$\frac{1}{\sqrt{y}}\left(\theta(y)-1\right)+\left(\frac{1}{\sqrt{y}}-1\right)$$

$$= \int_1^\infty y^{\frac{1-s}{2}}\,\frac{\sqrt{y}}{\sqrt{y}}\left(\frac{\theta(y)-1}{2}\right)\frac{dy}{y}+\int_1^\infty y^{\frac{1-s}{2}}\left(\frac{\frac{\sqrt{y}}{\sqrt{y}}-1}{2}\right)\frac{dy}{y}$$

$(1-s)$?!

Symmetric w/ $s \longleftrightarrow 1-s$

$$\overset{!?!}{=} \int_1^\infty y^{\frac{1-s}{2}} \frac{\Theta(y)-1}{2}\frac{dy}{y} + \boxed{\int_1^\infty y^{-s/2}\left(\frac{\sqrt{y}-1}{2}\right)\frac{dy}{y}}$$

✓ :')

not entire

(?!)

$s \longleftrightarrow 1-s$ ✓ :')

For $Re(s)>1$ (check)

is explicitly

evaluatable !

$$-\frac{s}{2}-\frac{1}{2}$$

$$= \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}+\frac{1}{2}-1}\,dy - \frac{1}{2}\int_1^\infty y^{-\frac{s}{2}-1}\,dy$$

$Re(s)>1$!

$$s-$$

$$= \cancel{\tfrac{1}{2}}\cdot\frac{1}{\frac{s}{\cancel{2}}-\frac{1}{\cancel{2}}} - \frac{1}{\cancel{2}}\,\frac{1}{\frac{s}{\cancel{2}}}$$ ✓ :')

?!

IOU

Supposed to have sym !

$$\cancel{\tfrac{1}{2}}\,\left(\frac{1}{s-1}+\frac{1}{-s}\right)$$:')