Phragmen-Lindelöf

\[ s \leq \gamma \]

& growth - order (Hadamard)

\[ \geq \text{Hadamard prod} \quad 1893 \]

Generalizes “Max Mod Principle”

For \( f \) holm on disk (et al.)

ext. cont. to bounday (\( \frac{1}{2} \))

Max If|If occurs on bounday. If \( h \) hom open \( \Rightarrow \) closed dish

Claim \( \forall s(1 + s + it) = \text{odd} \)

\[ s > 0 \]

& \( \exists (-s + it) = \text{poly growth} \)

\[ \Rightarrow \exists \text{ poly growth in} \]

\[ -s < \text{Re}(s) < 1 + s \quad (\text{eg., in crit. strip}) \]
Need that to know \[ \mathcal{F}(s) = e^{as+b} \prod \left(1 - \frac{s}{\rho} \right) e^{s\rho} \]
\[ s(1-s) \]
\[ \approx e^{as+b} \prod \left(1 - \frac{s}{\rho} \right) \]
\[ \text{Rearranged due to} \]
\[ \rho \to 1 - \rho \]
\[ \mathcal{F}(1-s) = \mathcal{F}(s), \]
so \[ \rho \to 0 \iff 1 - \rho \to 0 \]
R's idea

Phr.-L. : Max mod. prime on \( \mathcal{V}(a) \)
is not quite true."

True, can map strip to lift to ing of \( \mathcal{F}(s) \), etc., but \( \mathcal{V}(a) \) is omitted.
So cannot reduce to MMP.
Consider $f(z) = e^{e^z}$ for $-\frac{\pi}{2} \leq \text{Re}(z) \leq \frac{\pi}{2}$.

Blow up in certain directions. $z = x + iy$, $e^z = e^x \cdot e^{iy} = e^x (\cos y + i\sin y)$.

So $f(z) = e^{e^z} = e^x \cdot (\cos y + i\sin y)$.

When $\cos y = 0$, $|e^z| = 1$, $f(z)$ is odd.

That is, $f$ is odd on edges.

On the other hand, at $\cos y = 1$ ($y = 0$), $|f(z)| = e^z$.

Not an extreme pathology.

Counterexample to "naive maxmod" for strip.
True Ph. - L. Then: On $-\frac{\pi}{2} \leq \text{Im}(z) \leq \frac{\pi}{2}$, for holomorphic \textbf{ext. to} $C^0$ \textbf{for an boundary},

$|f(z)| \leq \frac{3C}{B \cdot |z|}$ \textbf{for some} $B < 1$.

If $|f| \leq 1$ \textbf{on boundary}, then $\frac{3C}{B} \leq 1$ in interior.

\textbf{E7t:} $\exists B < 1$ \& $C$ \textbf{st.} 

$|f(z)| \leq \frac{3C}{B} e^{|z|}$. $\ldots$

Given that $B$ \& $C$, more $B$ chosen to $1$, etc.

\(\text{Cont.}\)

E.g., if $f$ \textbf{is entire with growth-order $\lambda$}, then Ph. - L. applies in any strip where $f$ \textbf{has no edges}.
If "f has grow order γ" is \( \lambda < \infty \) \\( \lambda \) 

\[ \text{then} \quad |f(z)| \leq 3C \cdot e^{\log z} \cdot e^{\frac{1}{2}z}, \quad \forall z > 0 \]

(not quite \( \leq C \cdot e^{\frac{1}{2}z} \))

\[ \Rightarrow f \text{ meets the hyp of Ph.-L.} \]

on any strip where its held on boundary

Variant 2: (for \( \gamma \))

For \( f \) of poly growth on strip

\( 0 \leq \text{Im} |z| \leq 1 \),

\[ |f(z)| \leq 3C \cdot (1 + |z|)^N, \quad \text{then} \]

\( \Delta f \) sat. hyp. of Ph.-L. \( (\neq \text{not too crazy growth}) \)

then \( f \) sat. same bound in interior.
On \( 0 \leq \text{Im}(z) \leq 1 \)

1. Consider \( \frac{f(z)}{(2 + 3)^N} \) \( (e > 0) \)

usual: apply Ph. 6. for each \( e > 0 \), slight cleverness \( \Delta + \theta \text{ the inf/min.} \)

set poly frontier

\[ n \]

2. \( \text{Apply to } \xi \)

Get crude not-crazy growth from integral presentation via \( \Theta \) .....

\[ s(1-s) \times \Gamma(\xi/2) \xi(1) = \left( \frac{1}{\xi-1} + \frac{1}{-\xi-1} + \int \frac{1}{y^{\xi/2} + y^{1-\xi/2}} \text{d}y \right) \]

odd vertically

add in abs. value in vertical by values on IR, \( \sqrt{...} \)
In $\text{Re}(s) \geq 1 + \delta$, $\zeta(s)$ held vertically.

$-\frac{1}{2}$ held vertically.

$\Gamma\left(\frac{3}{2}\right) \neq \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}$

$\zeta(1-s) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \Gamma\left(\frac{1-s}{2}\right) \zeta(s)$

held on vertical strips

$\text{Re}(s) \geq 1 + \delta$
\( \zeta(-\frac{1}{2} - it) = \frac{\pi^{\frac{3}{2} + it}}{\Gamma\left(\frac{3}{2} + it\right)} \frac{\Gamma\left(\frac{3}{2} + it\right)}{\Gamma\left(-\frac{1}{2} + it\right) \Gamma\left(-\frac{1}{2}\right)} \)

\( \text{Use asympt. } \frac{\Gamma(s)}{\Gamma(s+a)} \approx \frac{1}{s^a} \)

\( \Gamma\left(\frac{3}{4} + it\right) = \Gamma(\frac{3}{4} + it) \), so

\[ \left| \Gamma\left(-\frac{1}{2} - it\right) \right| = \Gamma\left(-\frac{1}{2} + it\right) \]

So the value is

\[ \frac{\Gamma\left(-\frac{1}{4} + it\right)}{\Gamma\left(\frac{3}{4} + it\right)} \]

\[ = \frac{\Gamma(s)}{\Gamma(s+1)} \approx \frac{1}{s} \text{ (Watson's Lemma)} \]

(See notes)
\[ |\xi(-\frac{1}{2} + it)| \approx \exp \left( \frac{3}{4} \right) \text{ odd} \]

Can (since \( \xi \) not crazy in interior of strip)

\( \xi \) in poly growth (\( \exp \leq 1 \) in strip)

\[ \text{anymp of } \Gamma(s), \text{ see that, yes} \]

\[ \zeta \text{ of growth order } \frac{1}{2} \]

A product desired by Riemann

IS CORRECT
Show $\zeta(s) \neq 0$ in $\text{Re}(s) > 1$.

Use Euler prod (arg in $\text{Re}(s) > 1$)

Take $s > 0$, & consider $\text{Re}(s) \geq 1+\delta$

\[
\zeta(s) = \prod_{\rho} \left(1 - \frac{1}{\rho^s}\right) \quad \text{for } \rho \in \text{RHS}
\]

\[
\left|\frac{1}{\zeta(s)}\right| = \prod \left|1 - \frac{1}{\rho^s}\right|
\]

\[
\leq \prod \left(1 + \frac{1}{\rho^{-\sigma}}\right) \quad (\sigma = \text{Re}(s))
\]

\[
= \gamma s \quad \text{(Sim. for Euler prod of $\zeta$ of Gaussian $\mathbb{Z}[i]$ as in Siegel)}
\]

Eulerian $\Rightarrow$ PID $\Rightarrow$ UFD