1908
Phragmén–Lindelöf : & ζ in crit. strip
& growth-order (Hadamard)

→ Hadamard prod. ! 1893

Generalizes "Max Mod Princ." → R.'s explicit fml (etc.) 1858 ☺

For f holo on disk (etc.)
ext. cont. to boundary (!) — f holo on open $\supseteq$ closed disk

Max $|f|$ occurs on boundary.

$$\text{Want: } \overset{\text{claim}}{\wedge}\ \zeta(1+\delta+it) = \text{bdd} \quad \delta>0$$

$$\& \quad \zeta(-\delta+it) = \text{poly growth}$$

$\overset{?!?}{\Longrightarrow}$ ζ poly growth in

$$-\delta < \mathrm{Re}(s) < 1+\delta$$

(e.g., in crit. strip.)

(away from pole @ $s=1$)

?!?

Need that to know ... ξ is order 1

Hadamard

to know $\underbrace{\xi(s)}_{s(1-s)} = e^{as+b} \prod_{\rho} \left(1-\frac{s}{\rho}\right) e^{s/\rho}$

$\simeq e^{a's+b'} \prod_{\rho} \left(1-\frac{s}{\rho}\right)$ grouped

rearranges

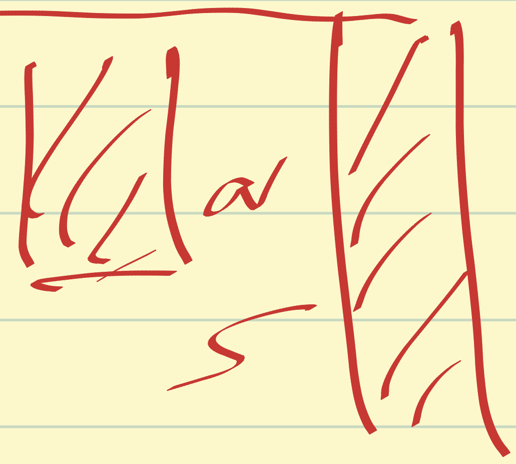
$\rho \leftrightarrow 1-\rho$

due to

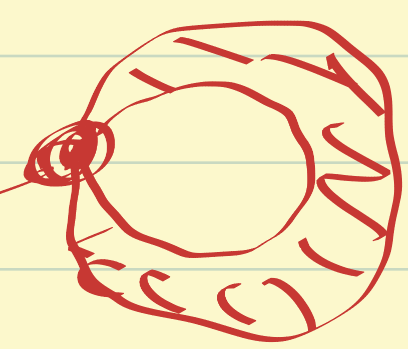
$\xi(1-s) = \xi(s)$,

so ρ a $0 \Leftrightarrow 1-\rho$ 0

R.'s idea

Phr.-L. : Max mod. princ. on 

is not quite true :(

True, can map strip to l.f.t. to 

img of ∞

etc., but w/ some pts on ∂ omitted.

So cannot reduce to MMP :(

$\exists$ (counter-) example :

Consider $f(z) = e^{(e^z)}$ on $-\frac{\pi}{2} \le \cancel{\mathrm{Re}}\,\mathrm{Im}(z) \le \frac{\pi}{2}$ $\mathrm{Im} = y$

Blow up in certain directions ✓ ☺

$z = x+iy$, $e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$$e^{(e^z)} = e^{e^x \cdot (\cos \cancel{x}\, y + i \sin y)}$$

$$= \underbrace{e^{e^x \cdot i \sin y}}_{|\cdot| = 1} \times \underbrace{e^{e^x \cdot \cos y}}_{\text{size of } e^{e^z}}$$

! when $\cos y = 0$, $|e^{e^z}| = 1$. = bdd

That is, f is bdd on edges

Otoh, at $\cos y = 1$ $(y=0)$, $|f(z)| = e^{e^x}$

= blows up

Not an extreme pathology.

→ Counter-example to "naive maxmod" for strip. ✓

True Ph.-L. Thm: On $-\frac{\pi}{2} \leq \mathrm{Im}(z) \leq \frac{\pi}{2}$, for holo fcn ext. to C^0 fcn on boundary,

$\boxed{\text{growth}}$ & $|f(z)| \leq \cancel{\exists C} \; \boxed{e^{e^{B\cdot|z|}}}$ for some $B < 1$,

$\Longrightarrow$

If $|f| \leq 1$ on boundary, then ≤ 1 in interior.

Eqt: $\exists\, B < 1$ & C s.t.

$$|f(z)| \leq C \cdot e^{e^{B\cdot|z|}} \ldots$$

Given that B & C, move B closer to 1, etc.

Cor E.g., if f is entire with Hadamard growth-order λ, then Ph.-L. applies in any strip where f bdd on edges;

$\underline{\text{Df}}$: "$\underbrace{f}_{\text{entire}}$ has groth order λ" is $\boxed{\lambda < \infty !}$

$$|f(z)| \leq {}^{\exists}C_\varepsilon \cdot e^{|z|^{\lambda+\varepsilon}}, \quad \underline{\forall \varepsilon > 0}$$

(<u>not</u> quite $\leq C \cdot e^{|z|} \ldots$)

$\Rightarrow$ f meets the hyp of Ph.-L. on any strip where it's bdd on boundary ✓ "!!"

<u>Variants</u>: (for ξ)

For $\underbrace{f}_{\text{entire}}$ of <u>poly growth</u> on (edges of a) strip

$$0 \leq \text{Im}(z) \leq 1,$$

$$|f(z)| \leq {}^{\exists}\underline{C} \cdot (1 + |z|)^{\exists \underline{N}}, \quad \text{then}$$

& f sat. hyp. of Ph.-L. (= not too crazy growth)

then f sat. <u>same bound</u> in <u>interior</u>.

On $0 \leq \mathrm{Im}(z) \leq 1$

Pf: Consider $\dfrac{f(z)}{(z+\varepsilon)^N}$ $(\varepsilon > 0)$

Usual: apply Ph.-L. for each $\varepsilon > 0$,

slight cleverness { & take inf/min. --- get poly growth ✓

//

§§ Apply to ζ:

Get crude not-crazy growth from integral presentation via θ ... :

$$\underbrace{s(1-s)}_{\text{bdd vertically}}\, \underbrace{\pi^{-s/2}\Gamma(s/2)\zeta(s)}_{} = s(1-s) \times \left(\underbrace{\frac{1}{s-1} + \frac{1}{-s}}_{\text{nice ✓}} + \underbrace{\int_1^\infty \left(y^{s/2} + y^{1-s/2}\right) \frac{\theta(y)-1}{2} \frac{dy}{y}}_{\text{bdd in abs. value in vertical lines by values on } \mathbb{R}.\ ✓} \right)$$

In $\mathrm{Re}(s) \geq \underline{1+\delta}$, $\zeta(s)$ bdd vertically.

$\pi^{-s/2}$ bdd vertically

$*$ $\underline{\Gamma(s/2)\ !?}$: $\dfrac{1}{\Gamma(s/2)}$ in $\left(\xi(s) = \xi(1-s), \dots\right)$

$$\zeta(1-s) = \frac{1}{\pi^{-\frac{(1-s)}{2}}\,\Gamma\left(\frac{1-s}{2}\right)}\,\pi^{-s/2}\,\Gamma(s/2)\,\underline{\zeta(s)}$$

bdd

$\underbrace{\mathrm{Re}(s) \geq 1+\delta}$ ✓ "

elem

bdd on vertical strips

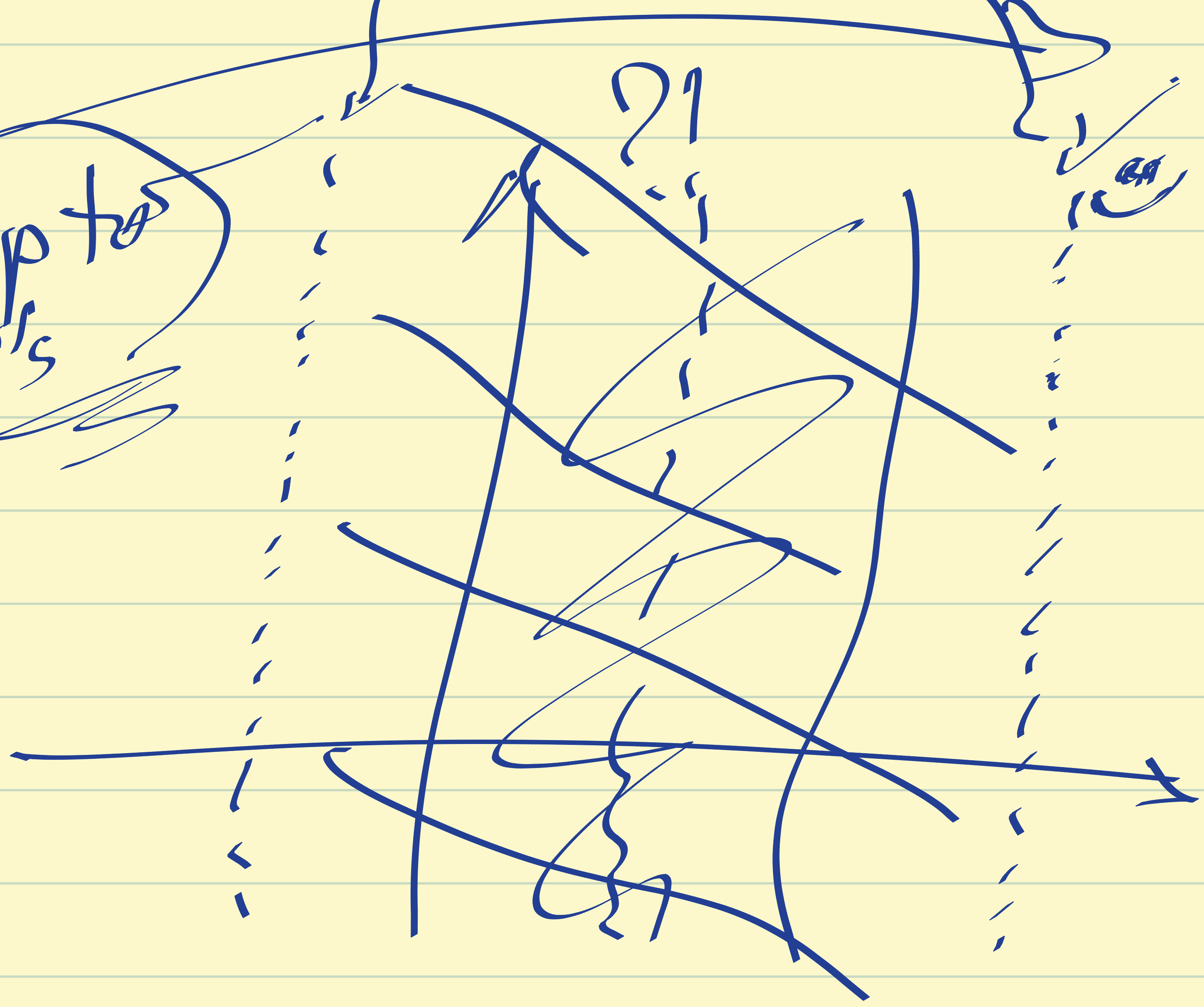
$$\frac{\Gamma(s/2)}{\Gamma\left(\frac{1-s}{2}\right)}$$

up to $\Gamma^{1/5}$

on $\mathrm{Re}(s) = 3/2$ (> 1)

ζ

convention



?! (Ph. L.)

$$\zeta\left(-\tfrac{1}{2}-it\right) = \underbrace{\frac{\pi^{-\left(\frac{3}{2}+it\right)}}{\pi^{-\left(\frac{1}{2}+it\right)}}}_{\text{bdd}} \underbrace{\frac{\Gamma\left(\frac{\frac{3}{2}+it}{2}\right)}{\Gamma\left(\frac{-\frac{1}{2}-it}{2}\right)}}_{?!} \underbrace{\zeta\left(\tfrac{3}{2}+it\right)}_{\text{bdd}}$$

Use asympt. $\dfrac{\Gamma(s)}{\Gamma(s+a)} \approx \dfrac{1}{s^a}$

& $\Gamma(\bar{s}) = \overline{\Gamma(s)}$, so

$$\left|\Gamma\left(\frac{-\frac{1}{2}-it}{2}\right)\right| = \Gamma\left(\frac{-\frac{1}{2}+it}{2}\right)$$

So abs value is $\dfrac{\Gamma\left(-\frac{1}{4}+\frac{it}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{it}{2}\right)}$ upside-down

see notes

$$= \frac{\Gamma(s)}{\Gamma(s+1)} \approx \text{"}\frac{1}{s}\text{"}$$

Watson's Lemma (easier than Stirling)

$$\Longrightarrow \left|\zeta\left(-\tfrac{1}{2}+it\right)\right| \approx |s| \cdot \underbrace{\left|\zeta\left(\tfrac{3}{2}+it\right)\right|}_{\text{bdd}}$$

Cor (since ζ not crazy in interior of strip)

ζ is poly growth ($\exp \leq 1$) in strips. ✓

+ asymp. of $\Gamma(s)$, see that, yes

ζ is of growth order 1 (Hadamard)

& product desired by Riemann

IS CORRECT

☺

M

⊗ Show $\zeta(s) \neq 0$ in $Re(s) > 1$.

Use Euler prod (cvgt in $Re(s) > 1$)

Take $\delta > 0$, & consider $Re(s) \geq 1+\delta$

✓ — $\left(\{s : Re(s) > 1\} = \bigcup_{\delta > 0} \{s : Re(s) \geq 1+\delta\} \right)$

:-)

$$\frac{1}{\zeta(s)} \overset{\text{Euler}}{=} \frac{1}{\prod_p \frac{1}{\left(1 - \frac{1}{p^{s}}\right)}} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^{s}}\right)$$

$$\left| \frac{1}{\zeta(s)} \right| = \prod \left| 1 - \frac{1}{p^{s}} \right|$$

$$\overset{!}{\leq} \prod \left(1 + \frac{1}{p^{\sigma}}\right) \qquad (\sigma = Re(s))$$

= you — :-)

(Sim. for Euler prod of ζ of Gaussian $\mathbb{Z}[i]$ as in hmwk —)

Euclidean ⇒ PID ⇒ UFD