25 Feb 2022

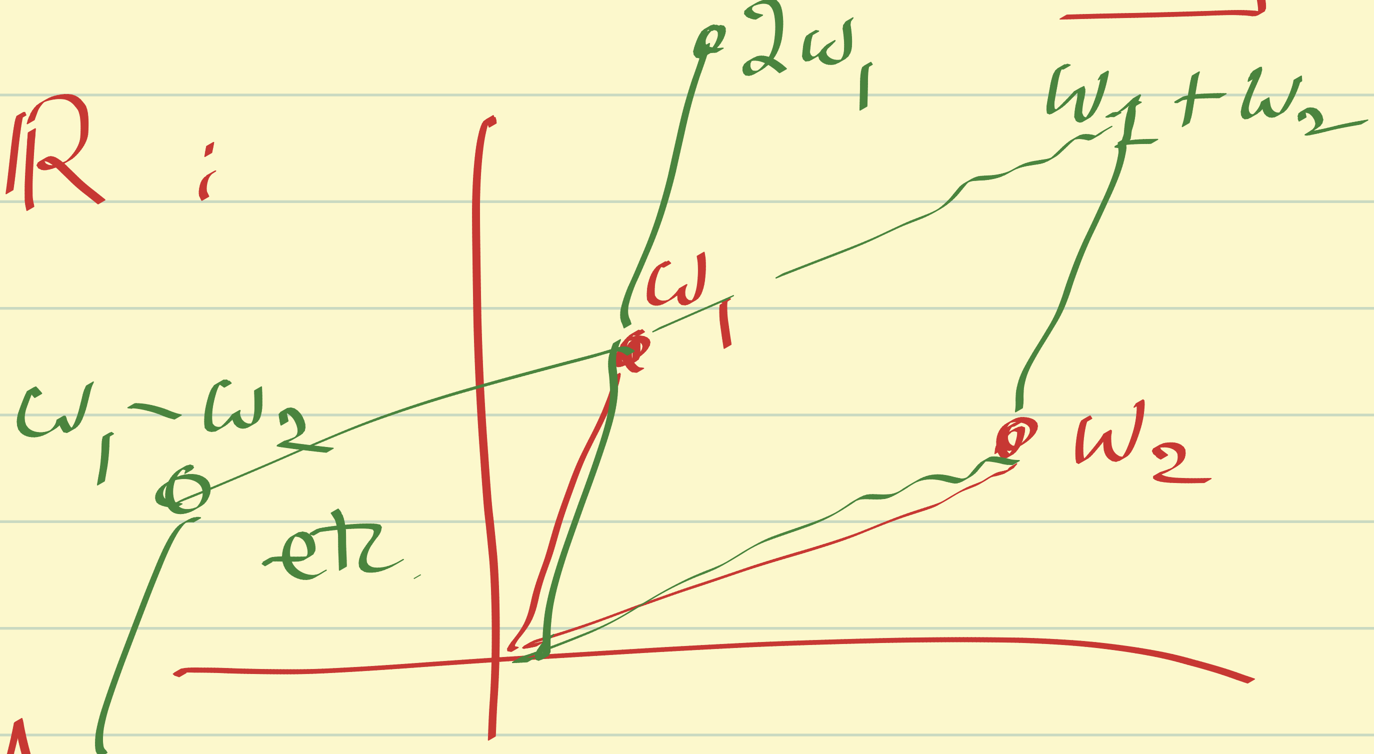
More in notes!

A doubly periodic fcn f is ~~mero~~ holo on $\mathbb{C}$

st. $\exists$ ω_1, ω_2 ("periods") st.

$$\boxed{f(z+\omega_1) = f(z) = f(z+\omega_2)}$$

lin. indept over $\mathbb{R}$:



Then, $\forall\ \lambda \in \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

= subgp of $\mathbb{C}$ gen'd by ω_1 & ω_2

$$f(z+\lambda) = f(z),$$

(Likewise, for singly-periodic e^z, etc.,

the subgp is singly-gen'd, like $\mathbb{Z} \cdot \frac{2\pi i}{\omega_1}$)

Q: What are discrete subgps of $\mathbb{C}$?

For holo/mero f, if $f(z+\lambda_1) = f(z+\lambda_2)$

$= \ldots = f(z+\lambda_n) = \ldots$

& λ_n's have acc pt., then f is const

by Id. Principle :)

Eg., $\omega_1 = 1$ & $\omega_2 = \sqrt{2}$ (not lin. indep't over $\mathbb{R}$)

$\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot \underbrace{\sqrt{2}}_{\text{irrat.}} \overset{!?}{\underset{\text{ex}}{=}}$ dense in $\mathbb{R}$

(pigeon-hole princ., ...)

If f is holo/mero & $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\omega$ – periodic

then f is const.

needs 2 gen's. Contrast: $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\frac{5}{3}$

= singly-gen'd (!) ex

&/or $\mathbb{Z}\cdot 15 + \mathbb{Z}\cdot 49$ = singly gen'd $\overset{?}{\cong} \mathbb{Z}$:)

"Periodic fcns" : **Q** : discrete (closed) subgroups

of $\mathbb{C}$ are ? $\rightsquigarrow \mathbb{R}^2$

$\boxed{\text{of } \mathbb{R}^n \text{ are ?}}$

A: Of form $\boxed{\mathbb{Z}\cdot\omega_1 + \dots + \mathbb{Z}\cdot\omega_m}$

$m \leq n$

& ω_i's are lin. indep. over $\mathbb{R}$

Cor: On $\mathbb{C}$, $\nexists$ triply-periodic fcns!

(except constants ...)

Lemma: entire (= holo, not mero) doubly-periodic f is const. → "?!" → rigidity

(Contrast singly-periodic: $\exists\ e^z, e^{2z}, e^{3z} \dots$)

$\mathbb{C} \simeq \mathbb{R}^2$

Let f be entire & $f(z+\omega_1) = f(z)$
$= f(z+\omega_2)$
w/ ω_1, ω_2 lin. ind. over $\mathbb{R}$

Lⁿ: Claim:

Every $w \in \mathbb{C}$ is (uniquely)
expressible as $w = \lambda + a\omega_1 + b\omega_2$
for $\exists\, \lambda \in \Lambda$ = lattice & $0 \leq \exists\, a < 1$, $0 \leq \exists\, b < 1$

max disc. subgp

$\exists\, m, n \in \mathbb{Z}$

$m\omega_1 + n\omega_2$

Pf: Since ω_1, ω_2 lin indep over $\mathbb{R}$, &, thus, are $\mathbb{R}$-basis of $\mathbb{C}$, given $w \in \mathbb{C}$

$$\exists\, A, B \in \mathbb{R} \text{ s.t. } w = A\omega_1 + B\omega_2$$

Let $A = m + a \quad m \in \mathbb{Z},\ 0 \leq a < 1$
$B = n + b \quad n \in \mathbb{Z},\ 0 \leq b < 1$

"$\mathbb{Z} \subseteq \mathbb{R}$"

Then $w = \underbrace{(m\omega_1 + n\omega_2)}_{\lambda \in \Lambda} + a\omega_1 + b\omega_2$

as desired. ✓

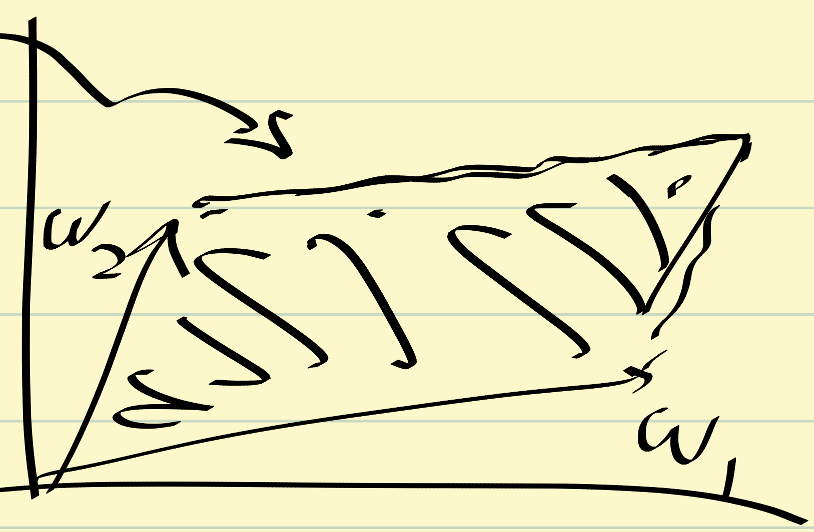
claim:

Pf of thm: if f doubly periodic & entire,

then bdd $\underbrace{\left(\xrightarrow[\text{Liouville}]{} \text{const.} \right)}_{\text{rigidity}}$

f is cont. on cpt. $\{ a\omega_1 + b\omega_2 : 0 \le a \le 1, \ 0 \le b \le 1 \}$ = closed parallelogram

So bdd on cpt



But, $\forall w \in \mathbb{C}$,

$w = \lambda + a\omega_1 + b\omega_2$, $\lambda \in \Lambda$

$$f(w) = f(\lambda + a\omega_1 + b\omega_2) \overset{!}{=} f(a\omega_1 + b\omega_2) = \text{bdd} \Longrightarrow \text{Const.}$$

!! ?!

rigidity

Aux: ooh: all ~~rank-one~~ discrete sbgps of $\mathbb{C}$ are "the same",

$\forall\, \omega, \omega' \neq 0, \quad \exists\, \alpha \in \mathbb{C}^\times$

s.t. $\alpha \cdot (\mathbb{Z} \cdot \omega_1) = \mathbb{Z} \cdot \omega_2 \quad \left(\because \alpha = \dfrac{\omega_2}{\omega_1}\right)$

otoh: $\exists$ continuum of "not the same" rank-two "lattices" in $\mathbb{C}$:

$\nexists$ holo $\mathbb{C} \xrightarrow{m} \mathbb{C}$ s.t. $m(\Lambda_1) = \Lambda_2$

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \cup\!| & & \cup\!| \\ \Lambda_1 & & \Lambda_2 \end{array}$$

!!!

Later: $\underbrace{\{\simeq\text{-classes}\}}_{\text{a "moduli space" for \{lattices\}}} \simeq SL_2(\mathbb{Z}) \backslash \mathfrak{h}$ — not cpt nearly cx mfld "

by l.f.t.'s ☺

nice that moduli space has reasonable physical sense

Make some doubly-periodic fcns

(Yes, inverting $\int_0^z \frac{dt}{\sqrt{1+t^4}}$ ...,)

cubic or quartic

$\sim$ inverting

$\int_0^x \frac{dt}{\sqrt{1+t^2}} \rightsquigarrow$ sine)

$\to$ Weierstraß (Eisenstein, Abel, Jacobi, Gauss...)

TeX: \wp = Weierstraß' "P" $\neq$ script / fraktur

$\wp \quad \wp \quad \wp$

Average/periodize/automorphize simpler fcn

Given Λ

"$f(z) = \sum_{\lambda \in \Lambda} \varphi(z+\lambda)$"

for convenient φ ??

Then (!) $f(z+\underset{\Lambda}{\overset{\in}{\mu}}) = \sum_{\lambda \in \Lambda} \varphi((z+\mu)+\lambda)$

$= \sum_{\lambda \in \Lambda} \varphi(z+\lambda) = f(z)$ ✓

Λ group: $\lambda \leftrightarrow \lambda - \mu$

$$\varphi(z) = \frac{1}{z^\ell}$$ (for cvgce away from poles)

$\ell = 1, 2, 3, \ldots ?$

Lemma: (yay!) $\displaystyle\sum_{\mathbb{Z}^n \ni \lambda} \frac{1}{|\lambda|^\ell}$ cvgt $\Longleftrightarrow$ $\ell > n$

$\frac{1}{z^2}$ fails — " , $\frac{1}{z^3}$ work ($\equiv$ converge)

Let

$$\wp'(z) = \underbrace{-2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}}_{\text{cvgt}\checkmark}$$

$\frac{d}{dz}$

W! $$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\neq 0\\ \lambda\in\Lambda}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

("convergence trick — ")

abs. cvgt ✓ "

unif. cvgts $\Longrightarrow$ holo/mero

$\wp' =$ "obvs." doubly periodic ✓ "

$\wp$ doubly periodic ?! — for Λ

$\wp' =$ periodic, integrate,

Fix $\lambda \in \Lambda$, $\underbrace{\wp(z) - \wp(z+\lambda)} = C_\lambda$??, $\forall \lambda \in \Lambda$

Use $\wp$ is even: $\wp(-z) = \wp(z)$

take $z = \lambda/2$: $C_\lambda = 0$, $\forall \lambda \in \Lambda$ ✓ "

Thm (W.) ~~Fix~~ For all Λ,

KdV $\equiv$ $\left\{ \wp'(z)^2 = 4\wp(z)^3 - \cancel{60}\, g_2 \wp(z) - \cancel{140}\, g_3 \right.$ dep on Λ

dep. on Λ $\left\{ \begin{array}{l} \text{w/} \quad g_2 = 60 \cdot \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4} \\ \\ g_3 = 140 \cdot \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6} \end{array} \right\}$ "Eisenstein" series — " "modular forms"