

28 Feb 2022

Can postpone exam!?! let me know

Noteworthy points (pre-elliptic form)

! $\zeta(s) \neq 0$ in $\text{Re}(s) > 1$:

// def'n

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

does not prove it —

until $\text{Re}(s)$ is soooo large that $\frac{1}{n^s}$ dominates
 ?!?

! Use (convenient!!) Euler product:

$$\zeta(s) \stackrel{!!!}{=} \prod_{\substack{p \text{ prime} \\ \text{Re}(s) > 1}} \frac{1}{1 - p^{-s}}$$

In that region, "take logs" (after taking abs
vals?!)

$$\log |\zeta(s)| = - \sum_p \underbrace{\log |1 - p^{-s}|}$$

elem. estimate by $\log(1-x) = -x - \frac{x^2}{2} - \dots$

Have : all summands $\lg |1-p^{-s}|$

are $\neq \pm\infty$ ($\sim$ factor $= \prod_{n=1}^{\infty} (1-p^{-ns})$?)

Familiar

$$\Delta = \prod_n (1+a_n) \text{ conv}$$

with $\underline{no} \ a_n = -1$

$$\approx \sum_n a_n$$

methodological

Here : tangible case : don't let specific details distract you!

less familiar, but still, ... !!!

$$\sum_{\alpha \in \mathbb{Z}[i], \alpha \neq 0} \frac{1}{|\alpha|^{2s}} \neq 0 \text{ in } \operatorname{Re}(s) > 1;$$

Resembles "Eisenstein series" arising in elliptic form; $\mathbb{Z}[i]$ = Gaussian integers

$$\cong \mathbb{Z} \cdot \underline{1} + \mathbb{Z} \cdot \underline{i}$$

(lin. indep. over $\mathbb{R}$ ✓)

AND

$$\mathbb{Z}[i] = \text{Euclidean} \Rightarrow \text{PID} \Rightarrow \text{UFD}$$

$$\& \mathbb{Z}[i]^{\times} = \{\pm 1, \pm i\}$$

! $\exists! \frac{4}{\leq} \rightsquigarrow \frac{1}{4}$

If $\alpha \cdot \beta = 1$ in $\mathbb{Q}(i)$ (= field of frac. of $\mathbb{Z}[i]$)

$$\text{then } N_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha\beta) = 1$$

// mult. of norm

$$N(\alpha) \cdot N(\beta)$$

$$\alpha, \beta \in \mathbb{Z}[i] \Rightarrow \text{norm} \in \mathbb{Z}$$

$$a, b \in \mathbb{Z} \Rightarrow \overline{\overline{a^2 + b^2}} \in \mathbb{Z} \checkmark \text{ "}$$

$$\parallel \geq 0$$
$$N(a+bi)$$

Solve " $a^2 + b^2 = 1$ " in $\mathbb{Z}$: $a = \pm 1$ & $b = 0$

$$\&/\vee \underbrace{a=0 \& b=\pm 1}$$

$$\pm 1 \& \pm i \checkmark \text{ "}$$

Recall, $\zeta(s) \stackrel{\text{usual}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$

$\stackrel{\checkmark}{=} \left(\frac{1}{\mathbb{Z}^\times} \right) \times \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{|n|^s}$
 card $\rightsquigarrow \mathbb{Q}$
 Suggest extensions $\checkmark$

$$\zeta_{\mathbb{Z}[i]}(s) \stackrel{!}{=} \frac{1}{|\mathbb{Z}[i]^\times|} \cdot \sum_{\substack{\alpha \in \mathbb{Z}[i] \\ \alpha \neq 0}} \frac{1}{|N\alpha|^s}$$

$\stackrel{!!!}{=} \prod_{\substack{\text{primes} \\ \omega = \text{"varpi"} \\ \text{mod} \\ \text{units}}} \frac{1}{1 - (N\omega)^{-s}}$
 $\stackrel{\text{Euler}}{\geq}$

$\{ \}$
 "primes" in $\mathbb{Z} = \text{pos. rep. for } 1 \pm p$ $\checkmark$
 "

$\Rightarrow \zeta(s) \neq 0$ for $\operatorname{Re}(s) > 1$ ✓
 Euler prod $\mathbb{Z}[i]$

+ cvge in !?! — notes / discussion

Attached to lattice $\Delta \cong \mathbb{Z}^2$ — $f_\Delta(z)$ or $f(z; \Delta)$
 $f(z)$ & $f(z)$: $\{ \}$
 ??

$$\frac{1}{z^2} + \sum_{0 \neq \lambda \in \Delta} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) \quad \text{cvge!?!}$$

for cvge

$$f'(z) = -2 \sum_{\lambda \in \Delta} \frac{1}{(z-\lambda)^3}$$

fix Δ

abs cget, mit an cpts ✓

Then $f'(z)^2 = 4f(z)^3 - g_2 f'(z) - g_3$

$g_2 = g_2(\Delta) = \sum_{0 \neq \lambda \in \Delta} \frac{1}{\lambda^4}$?
 $g_3 = g_3(\Delta) = \sum_{0 \neq \lambda \in \Delta} \frac{1}{\lambda^6}$?

$$\left(\sin' z \right)^2 = 1 - \left(\sin z \right)^2$$

$$x = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$$

$$g_2 = 60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4}$$

$$g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

ask facts

universal

over Λ 's ! " ! "

pf idea (see notes)

use ~~by~~ periodicity ! !

(1) look at Laurent expn's @ $z=0$

(2) use fact that entire ell fn is const.

→ cancel pole(s) @ 0, cancelled all ! "

$$\left(\sim \frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \right) \quad ! "$$

Thm 2 : The field of elliptic fcn with
period lattice Δ is $\mathbb{C}(p_\Delta, p'_\Delta)$
 (pf: notes —) rat'l exps in p & p'

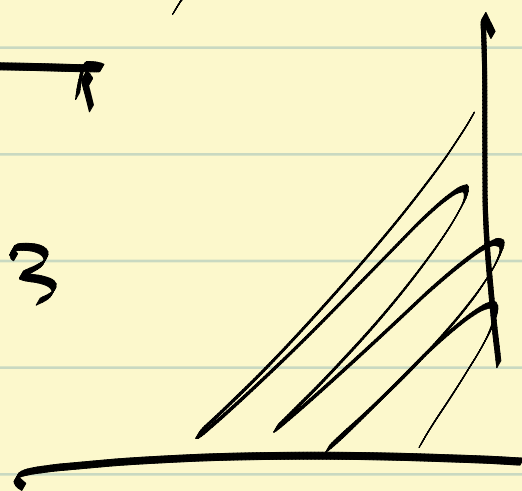
+ subfield of even ell. fcn ($f(-z) = f(z)$)
 $= \mathbb{C}(p_\Delta)$

Gal. theory

$\mathbb{C}(p_\Delta, p'_\Delta)$

/ 2 $\text{gal gr} \ni f(z)$
 $\downarrow$
 $\mathbb{C}(p_\Delta)$ $f(-z)$

$$p'_\Delta = \sqrt[3]{4p_\Delta^3 - g_2p_\Delta - g_3}$$



Fix Δ .

Eg., f'' is even

$$\underbrace{f(-z)}_{d/dz} = \underbrace{f(z)}_{\text{even}} \checkmark$$

$$\underbrace{f'(-z)}_{\text{odd}} = - \underbrace{f'(z)}_{\text{odd}}$$

$$f''(z) = f''(z)$$

Thm

$\exists$ rational expr for f''
in terms of f ✓

Simpler

$$\frac{d}{dz}$$

$$f'^2 = 4f^3 - g_2f - g_3$$

$$\cancel{2f'} f'' = 12f^2 \cdot \cancel{f'} - g_2 \cancel{f'}$$

$$2f'' = 12f^2 - g_2$$

✓

§§ "Lattices" : $\exists$ known ex : simplest case

Show discrete subgp of $\mathbb{R}$ is

(either $\Gamma = \{0\}$ or) $\Gamma = \mathbb{Z} \cdot x_0$

for $\exists x_0 \neq 0$.

Discussion : first need to know/see/show
 Γ is also closed :

$\exists$ discrete, not-closed subsets :

$$\left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$$

each has inv

disj from others : discrete —

— ops, but 0 is a limit pt, so
~~the~~ set is not closed —

(~ complications)

Granting closedness (see notes —)

} very general
} "abstract"

0th $\Gamma = \{0\}$ ✓

1st $\Gamma \neq \{0\}$. For $x \in \Gamma$, $-x \in \Gamma$, so $\exists$

$$0 < \underline{x} \in \Gamma$$

let $x_0 = \inf_{0 < x \in \Gamma} x$. Claim $x_0 \neq 0$:

$$0 < x \in \Gamma$$

$\underbrace{M}_0$ has ubd U at

discreteness $U \cap \Gamma = \{0\}$ ✓

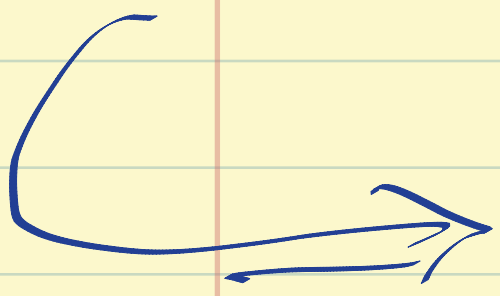
Then (closedness!) $0 < \underline{x_0} \in \Gamma$.

Arch. prop. of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $x = \exists nx_0 + r$

$$0 \leq r < x_0$$

Exp. for $x \in \Gamma$. If $0 < r$, then

$$0 < r = \underbrace{x - nx_0}_{\in \Gamma} < \underbrace{x_0}_{\#}$$



$$r=0, \quad \delta \quad x = nx_0$$

$\cap$

Γ

|

$i\omega x$

$x > 0$

$\cap$

Γ

