28 Feb 2022

Can postpone exam !?! let me know —

Noteworthy points (pre- elliptic fcn—)

! $\zeta(s) \neq 0$ in $\mathrm{Re}(s) > 1$:

$$\Big\| \text{defn}$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^s}}$$

does not prove it —

until $\mathrm{Re}(s)$ is soooo large that $\frac{1}{1^s}$ dominates

?!?

! Use (convergent!!) Euler product:

$$\zeta(s) \underset{\mathrm{Re}(s)>1}{\overset{!!!}{=\!=}} \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

In that region, "take logs" (after taking abs vals ?!)

$$\log|\zeta(s)| = -\sum_p \underbrace{\log|1-p^{-s}|}_{\text{elem. estimate by } \log(1-x) = -x - \frac{x^2}{2} - \dots}$$

Have: all summands $\log|1-p^{-s}|$

are $\neq \pm\infty$ ($\sim$ factor $\underset{\equiv}{=} 0$ (or ∞)?! :-))

Familiar $\delta = \prod_n (1+a_n)$ cvge

with $\underline{\text{no}}$ $a_n = -1$

$\underset{\exists \text{ pf} \sim}{\approx} \sum_n a_n$

methodological

Here: tangible case: don't let specific details distract you!

~~Less familiar, but, still, ... ! ! !~~

$\rightarrow \frac{1}{4} \sum_{\substack{\alpha \in \mathbb{Z}[i] \\ \not\circ}} \frac{1}{|\alpha|^{2s}} \neq 0$ in $\mathrm{Re}(s) > 1$:

?!

Resembles "Eisenstein series" arising in elliptic fns; $\mathbb{Z}[i]$ = Gaussian integers

$\underset{=}{\text{is}}$ $\mathbb{Z}\cdot\underset{=}{1} + \mathbb{Z}\cdot\underset{=}{i}$

lin. indep. over $\mathbb{R}$ ✓

AND $\mathbb{Z}[i]$ = Euclidean $\Longrightarrow$ PID $\Longrightarrow$ UFD

& $\mathbb{Z}[i]^\times = \underbrace{\{\pm 1, \pm i\}}_{\exists!\ 4 \ \rightsquigarrow\ \frac{1}{4}}$

(!)

If $\alpha \cdot \beta = 1$ in $\mathbb{Q}(i)$ (= field of frac. of $\mathbb{Z}[i]$)

then $N_{\mathbb{Q}(i)/\mathbb{Q}}(\alpha\beta) = 1$

|| mult. of norm

$N(\alpha) \cdot N(\beta)$

$\alpha, \beta \in \mathbb{Z}[i] \Longrightarrow$ norm $\in \mathbb{Z}$

$a, b \in \mathbb{Z} \Longrightarrow a^2 + b^2 \in \mathbb{Z}$ ✓ :)

$a^2+b^2 = N(a+bi) \geq 0$

Solve "$a^2+b^2=1$" in $\mathbb{Z}$: $a = \pm 1$ & $b = 0$

&/or $\underbrace{a = 0 \ \& \ b = \pm 1}$

± 1 & $\pm i$ ✓ :)

Recall, $\zeta(s) \overset{\text{usual}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$

$$\stackrel{\checkmark}{=} \frac{1}{\underset{2}{\underbrace{|\mathbb{Z}^\times|}_{\text{card}}}} \times \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{|n|^s} \qquad \subset \mathbb{Q}$$

Suggest extensions "✓

$$\zeta_{\mathbb{Z}[i]}(s) \stackrel{!}{=} \frac{1}{|\mathbb{Z}[i]^\times|} \cdot \sum_{0 \neq \alpha \in \mathbb{Z}[i]} \frac{1}{|N\alpha|^s}$$

!

$$\overset{!!!}{\underset{\text{Euler}}{=}} \prod_{\substack{\text{primes} \\ \varpi = \text{"varpi"} \\ \text{mod} \\ \text{units}}} \frac{1}{1-(N\varpi)^{-s}}$$

$\wr\|$

"primes" in $\mathbb{Z}$ = pos. rep. for $\{\pm p\}$ "✓

$\rightsquigarrow \Longrightarrow \zeta_{\mathbb{Z}[i]}(s) \neq 0$ for $Re(s) > 1$ ✓

Euler prod + cvgce!?! — notes/discussion

Attached to lattice $\underline{\Lambda}$ — "$\wp_\Lambda(z)$ or $\wp(z;\Lambda)$??"

$\underline{\wp(z)}$ & $\wp'(z)$:

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \underbrace{\frac{1}{\lambda^2}}_{\text{for cvgce}} \right) \text{ — } \underline{\text{cvgce!?!}}$$

$$\wp'(z) = -2 \underbrace{\sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3}}_{\text{abs cvgt, unif on cpts ✓}}$$

Fix Λ

$\underline{\text{Thm}}$ $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$?!

$g_k = g_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^{2k}}$??

$\longleftrightarrow \quad \underbrace{(\sin' z)^2} = 1 - \underbrace{(\sin z)^2}$

$$x = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}$$

?!

$$g_2 = \underbrace{60} \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4}$$

$$g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

artifacts

!:)

straightforward!

universal are Λ's !:)

pf idea (see notes):

use ~~by~~ periodicity ?!

(1) look at Laurent expn's @ $z=0$

(2) use fact that entire ell fcn is const.

→ Cancel pole(s) @ 0, cancelled all ✓

$$\left(\sim \frac{\pi^2}{\sin^2 \pi z} = \sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2} \right)$$

:)

Thm 2: The **field** of elliptic fcns with period lattice Λ is $\underbrace{\mathbb{C}(\wp_\Lambda, \wp'_\Lambda)}_{\text{rat'l exprs in } \wp \,\&\, \wp'}$

(pf: notes ___)

+ subfield of **even** ell. fcns $(f(-z) = f(z))$

$= \mathbb{C}(\wp_\Lambda)$

Gal. theory

$$\begin{array}{c} \mathbb{C}(\wp_\Lambda, \wp'_\Lambda) \\ \Big| \, 2 \\ \mathbb{C}(\wp_\Lambda) \end{array} \qquad \text{gal gp} \ni f(z) \mapsto f(-z)$$

$$\wp'_\Lambda = \text{``}\pm\text{''}\sqrt{4\wp_\Lambda^3 - g_2\wp_\Lambda - g_3}$$

Fix Λ.

Eg., $\wp''$ is even $\left(\underbrace{\wp(-z) = \wp(z)}_{\text{even}} \checkmark \right.$

d/dz

$$\underbrace{\wp'(-z) = -\wp'(z)}_{\text{odd}}$$

$$\wp''(z) = \wp''(z)$$

$\Downarrow$ Thm

$\exists$ rational expn for $\wp''$ in terms of $\wp$ ✓

Simpler : $\frac{d}{dz}$: $\wp'^{2} = 4\wp^3 - g_2\wp - g_3$

$\Downarrow$

$$2\cancel{\wp'}\wp'' = 12\wp^2 \cdot \cancel{\wp'} - g_2\cancel{\wp'}$$

$$2\wp'' = 12\wp^2 - g_2$$

✓

"

§§ "Lattices" : ∃ known ex : simplest case

Show discrete subgp Γ of $\mathbb{R}$ is

(either $\Gamma = \{0\}$ or) $\Gamma = \mathbb{Z} \cdot x_0$

for $\exists x_0 \neq 0$.

Discussion : first need to know/see/show

Γ is also closed :

∃ discrete, not-closed subsets :

$$\left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \right\}$$

each has nbd

disj from others ; discrete —

— oops, but 0 is a limit pt, so ~~A~~ set is not closed —

(~ complications)

Granting closedness (see notes ___)

{ very general "abstract"

0^{th} If $\Gamma = \{0\}$ ✓

1^{st} $\Gamma \neq \{0\}$. For $x \in \Gamma$, $-x \in \Gamma$, so $\exists$

$$0 < x \in \Gamma$$

Let $x_0 = \inf_{0 < x \in \Gamma} x$. Claim $x_0 \neq 0$:

$\underbrace{\text{If } 0}_{\text{discreteness}}$ has nbd U s.t. $U \cap \Gamma = \{0\}$ ✓ ☺

Then (closedness!) $0 < x_0 \in \Gamma$.

Arch. prop. of $\mathbb{R}$, $\forall x \in \mathbb{R}$, $x \overset{\exists}{=} n x_0 + r$

$$0 \leq r < x_0$$

Esp. for $x \in \Gamma$. If $0 < r$, then

$$0 < r = \underbrace{x - n x_0}_{\in \Gamma} \underbrace{<}_{\#} x_0$$

$r = 0$, & $x = n x_0$

$\Updownarrow$ $\quad$ $|$

iy_2

$x > 0$

$\Updownarrow$