∃ Exam 06 this Fri (honor system, email ....)

02 Mar 2022

If you want delay (+ examples!) TELL ME ...

Next week = spring break! :)

For given $\boxed{\text{lattice } \Lambda}$ $\forall$, $\wp$ & $\wp'$ relations & consequences:

\wp

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

$$\boxed{\wp'(z)^2 = 4\wp(z)^3 - g_2 \wp(z) - g_3}$$

$$g_2 = 60 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^4}$$

$$g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^6}$$

"attached to" lattice

|||

"fcns of lattice"

" = a thing"

(pf = notes) : look at Laurent coefs of

$(\wp')^2$ & $\wp^3$ & $\wp$, etc. @ 0 ——

& after arranging to cancel polar part

(& const ~ g_3)

what remains is entire doubly-per. fcn

const $\overset{\text{Liouville}}{\Longleftarrow}$ (~ Λ)

& $_{\sim}$ = 0 @ 0 $\Rightarrow$ it's 0 ✓

∃ computing ½ — ∃ straightforward description but messy to execute

Car(s) Many More relations among

(temp. not'n, "e" for "Eisenstein")

$$e_{2k}\left(=e_{2k}(\Lambda)\right)=\sum_{0\neq\lambda\in\Lambda}\frac{1}{\lambda^{2k}}$$

"$E_{2k}(\Lambda)$" — ∃ conventions normalization

abs. cvgt for $2k \geq 4$

($\lambda \leftrightarrow -\lambda$ . . . .

& $\not\equiv 0$ for $2k \in 2\mathbb{Z}$)

As in pf of W's cubic equation, explicate Laurent expn of $\wp$ @ 0 !

(not completely obvs that $\wp$ is Λ-per, tho' $\wp'$ is ... need lemma ✓)

Fix Λ, ☺

$$\boxed{\wp(z) = \underbrace{\frac{1}{z^2}}_{\substack{\text{polar part} \\ @\, 0}} + \underbrace{\sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)}_{\text{holo } @\, 0 \checkmark}}$$

☺

So $\wp(z) - \frac{1}{z^2}$ = holo @ $z = 0$

= p.s. expn

(which, when added to $\frac{1}{z^2}$, gives Laurent expn ✓)

Uniqueness of p.s. & of Laurent expn !

No matter trick, answer is same.

Example of "self-deception" (?!)

$$\sum_{n\in\mathbb{Z}} \frac{1}{z^n} = \left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right) + \left(z+z^2+z^3+\dots\right)$$

$$= \frac{1}{1-\frac{1}{z}} + \frac{z}{1-z}$$

$$= \frac{z}{z-1} + \frac{z}{1-z} = 0$$

?!

$\hookrightarrow$ Can compute p.s. coefs of $\wp(z) - \frac{1}{z^2}$ @ 0 by Taylor–Maclaurin (not Cauchy integral fmla)

$\equiv$ n^{th} p.s. coef is $\dfrac{\left(\wp(z) - \frac{1}{z^2}\right)^{(n)}\Big|_{z=0}}{n!}$?! ✓ ☺

+ $\wp(z) - \frac{1}{z^2}$ is even, so
odd-order coefs are 0 ...

$\left(\wp(z) - \frac{1}{z^2}\right)^{(2n)}\Big|_{z=0} =$ ($n=0$, it's 0: const term is 0 ✓)

$0 < 2n \in 2\mathbb{Z}$, ?!

$$\sum_{0 \neq \lambda \in \Lambda} \frac{(-2)(-3)(-4)\cdots(-2n-1)}{(z-\lambda)^{2n+2}}\Bigg|_{z=0}$$

convergence $\left(\frac{1}{\lambda^6}\right)$

$\diagup (2n)!$

(modulo details)

$$\wp(z) = \frac{1}{z^2} + 0 + \underline{3e_4 z^2 + 5e_6 z^4 + 7e_8 z^6 + \dots}$$

const.

& odd-order terms $= 0$

uniform in Λ

universal — trivial relations give

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3$$

$g_2 = 60e_4$, $g_3 = 140e_6$

($\equiv$ needs no info about e_{2n}'s)

Entails ∞-many relations among e_{2n}'s!

In fact, e_4 & e_6 determine all others!

Eg., diff W-rel'n :

$$2\cancel{\wp'}\wp'' = 12\wp^2\cancel{\wp'} - g_2\cancel{\wp'}$$

$$\wp'' = 6\wp^2 - \frac{g_2}{2}$$

$\frac{d}{dz}$: $\boxed{\wp''' = 12\wp\wp'}$ (g's gone !!)

expand Laurent exp. &

get recursion for $e_{2k>4,6}$

in terms of e_4, e_6 !

(Not idle : often e_{2k}'s are values of

L-fcns - - - ..... : values of

fam. L-fcns dep. on $<\infty$-many

"new" numbers (transcendental?!)

Change of pace: show a discrete* subgroup Γ of a top. gp* G is top.! closed

For subset X of top sp. Y to be discrete is $\forall x \in X$, $\exists$ open nbd U of x in Y

st. $X \cap U = \{x\}$.

Eg., $\{\frac{1}{n} : n = 1, 2, 3, \ldots\}$ is discrete in $\mathbb{R}$.

— NOT CLOSED (in $\mathbb{R}$); 0 ($\notin$ set) is a limit pt.

not a pathology

G a gp is top gp when

$$\begin{cases} G \times G \to G \text{ by } \underset{\text{op.}}{\underline{\text{mult}}} \text{ is cont.} \\ \& \\ G \to G \text{ by } \underline{\text{inverse}} \text{ is cont.} \end{cases}$$

$+$ (?!?) G is loc. cpt & Hausdorff (?!?)

~~loc. cpt~~ ?!?

beware!!!

An inf. dim. Hsp is not "a top-gp" ?!?

"top. v.s. ....."

(all metric sp's are H. ✓ ")

Y top sp is "$\underline{\underline{H.}}$"

$|||$

$\forall\, y_1 \neq y_2$ in Y, $\exists$ open nbds $\mathcal{U}_1 \ni y_1$ & $\mathcal{U}_2 \ni y_2$ st

$$\mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset \quad ✓$$

Silly-tiny example of (non-metric!)

top sp not H.

Y is Y + a set of subsets of Y, "open sets"

st $\emptyset$ is open, Y is open,

arb $\cup$'s of opens are open

finite $\cap$'s " " " "

why this is good / useful?

long story ✓ ☺

$Y = \{0, 1\}$ & opens are Y & $\emptyset$

"indiscrete top" ?! : illustrates scope of

ADD ADJECTIVES " defn / axioms

$\underline{\text{H}}$: discrete $\underline{\text{subgp}}$ Γ of top gp* G

esp. loc. cpt!

is (top) closed:

top clos.

Let $\underline{g \in G}$, $\underline{g \notin \Gamma}$, but $\underline{g \in \overline{\Gamma}}$. $g =$ acc. pt.

Then $\forall$ open $\mathcal{U} \ni 1_G$, $g \cdot \mathcal{U}$ is open $\ni g$,

& $\exists\, \gamma_1 \in \Gamma \cap g\mathcal{U}$

& $\gamma_2 \in \Gamma$, $\gamma_2 \neq \gamma_1$ ($\exists \infty$ many)

So $\underbrace{\gamma_1^{-1}\gamma_2}_{\overset{?}{\in}\,\Gamma} \in (g\mathcal{U})^{-1} \cdot (g\mathcal{U})$

$= \mathcal{U}^{-1} g^{-1} g \mathcal{U} = \mathcal{U}^{-1} \cdot \mathcal{U}$;

$\mathcal{U}^{-1} = \{u^{-1} : u \in \mathcal{U}\}$

Cont. of mult & inverse at id. $\in G$ $\equiv$ ex

$\forall$ open $V \ni 1_G$, $\exists$ open $U \ni 1$

s.t. $U^{-1} \cdot U \subseteq V$.

choices of $\gamma_1, \gamma_2 \in \Gamma$ depend on V, U

That is, $\forall$ nbds $V \ni 1_G$, $\exists$

$\underline{\gamma_1^{-1}\gamma_2} \neq 1$ in V. $\Longrightarrow$ $\#$ 1_G has

γ_1, γ_2 not equal

open nbd V s.t.

$V \cap \Gamma = \{1_G\}$

discreteness.

$\#$

$\underline{\text{So}}$ cannot be $\exists g \in \bar{\Gamma}$ but

$g \notin \Gamma$.

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