21 Mar 2022

(Ell. fcns ⇝) Ell. modular forms  14, 14a, 14b

parameter space
=
moduli space for elliptic curves

recall ?!

$\{\Lambda \text{ mod dilations } \Lambda \to c\cdot\Lambda\}_{c\in\mathbb{C}^\times}$

$\approx SL_2(\mathbb{Z})\backslash\mathfrak{h}$

$\mathbb{Z}\cdot z+\mathbb{Z}\cdot 1$

Shared

$\mathbb{C}/\Lambda \approx \mathbb{C}/\Lambda'$

$\Updownarrow$

$\Lambda' = {}^{\exists}c\Lambda$

$\wp_\Lambda$ ~ arc length of ellipses / lemniscates ~ $\int_0^x \frac{dt}{\sqrt{1+t^3}}$ versus $1+t^2$

⇝ trig integrals

INVERT:

doubly periodic fcns on $\mathbb{C}$

+

Weierstrass: converse: $\Lambda$-periodic fcns on $\mathbb{C}$ ⇝ curve

$\wp'^2_\Lambda = 4\wp^3_\Lambda - g_{2,\Lambda}\wp_\Lambda - g_{3,\Lambda}$

not ellipse ~

but is cubic curve

"elliptic curves"

$y^2 =$ cubic in $x$

$SL_2(\mathbb{Z}) \backslash \mathbb{H}$ ~ upper ½-plane } Granted ✓ "

$SL_2(\mathbb{Z})$ : $2\times 2$ mces, $\mathbb{Z}$ entries, $\det = 1$

$\mathbb{H}$ : by l.f.t. / Möbius } in $\mathbb{H}$ & [moduli space] = $SL_2(\mathbb{Z}) \backslash \mathbb{H}$

§§ — What does $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ look like? (see 14 appendix)

"Turns out" $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (ers) generate $SL_2(\mathbb{Z})$ !

(not elem! Pf ~ action on $\mathbb{H}$ & ∞)

// Given $\gamma \in \Gamma \subseteq SL_2(\mathbb{Z})$, ∃ numerical algo to express $\gamma \approx$ "word" in $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ & $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

but not a fml : "word" depends heavily on no. th. aspects of entries of $\gamma$ —

≡ continued-fraction expn

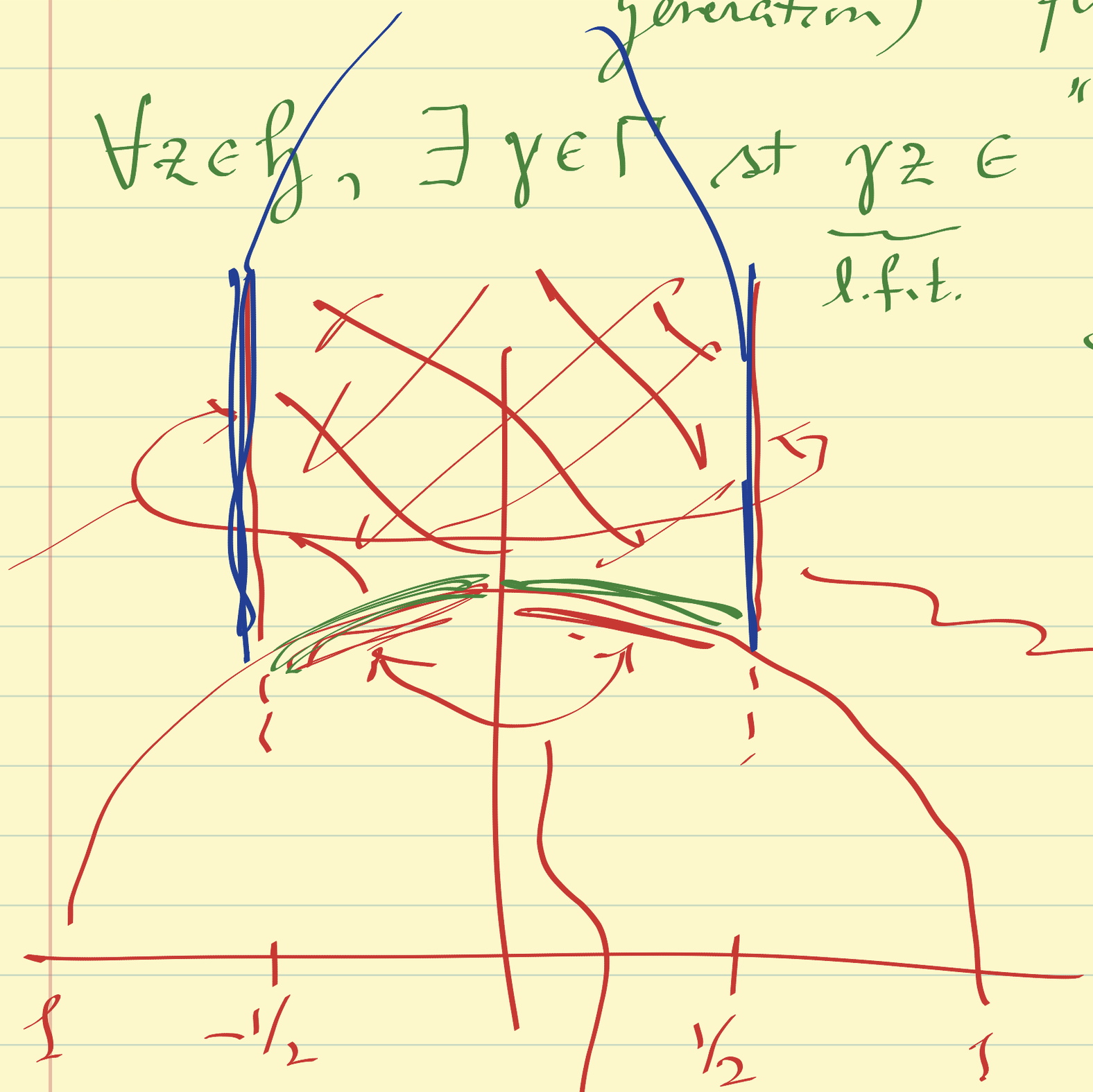
→ use existential fact that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ gen (not fml)

Turns out (by same "algo" that proves the generation) that

$\forall z \in \mathbb{H}$, $\exists \gamma \in \Gamma$ st $\gamma z \in$ l.f.t.

"the standard fundamental domain"

not canonical



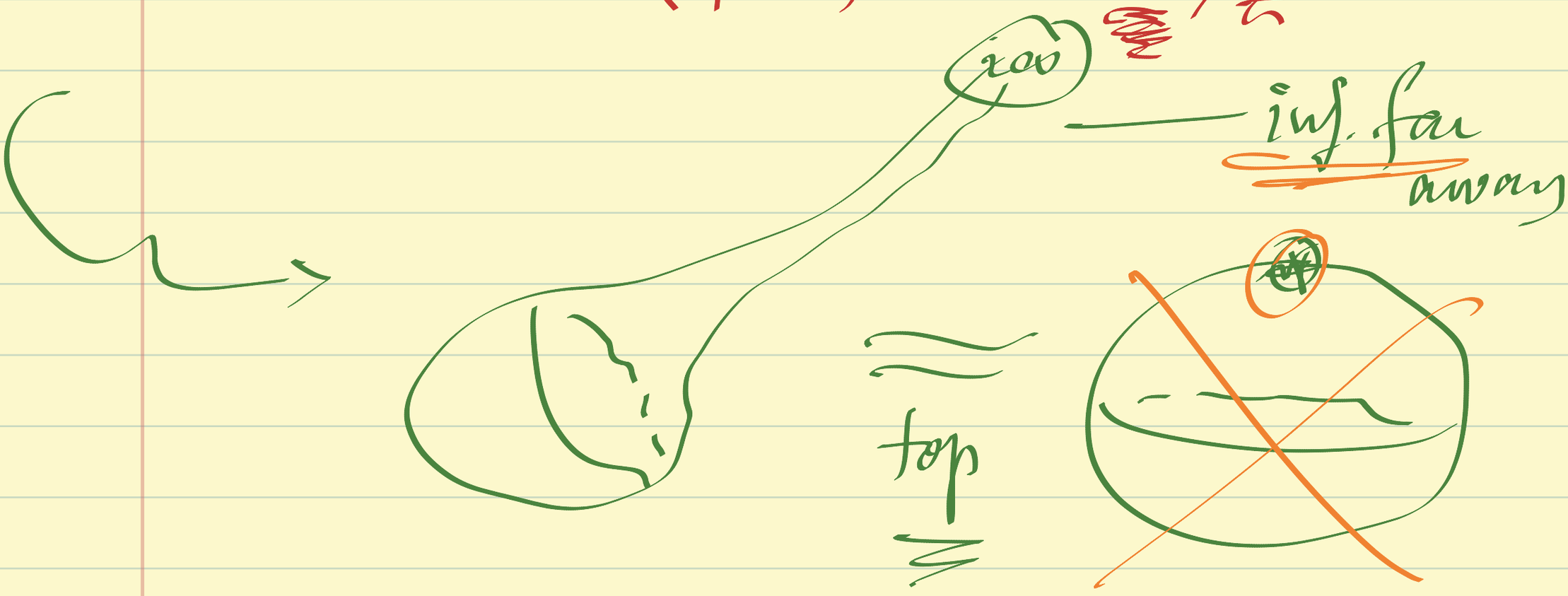

+ details @ edges

by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

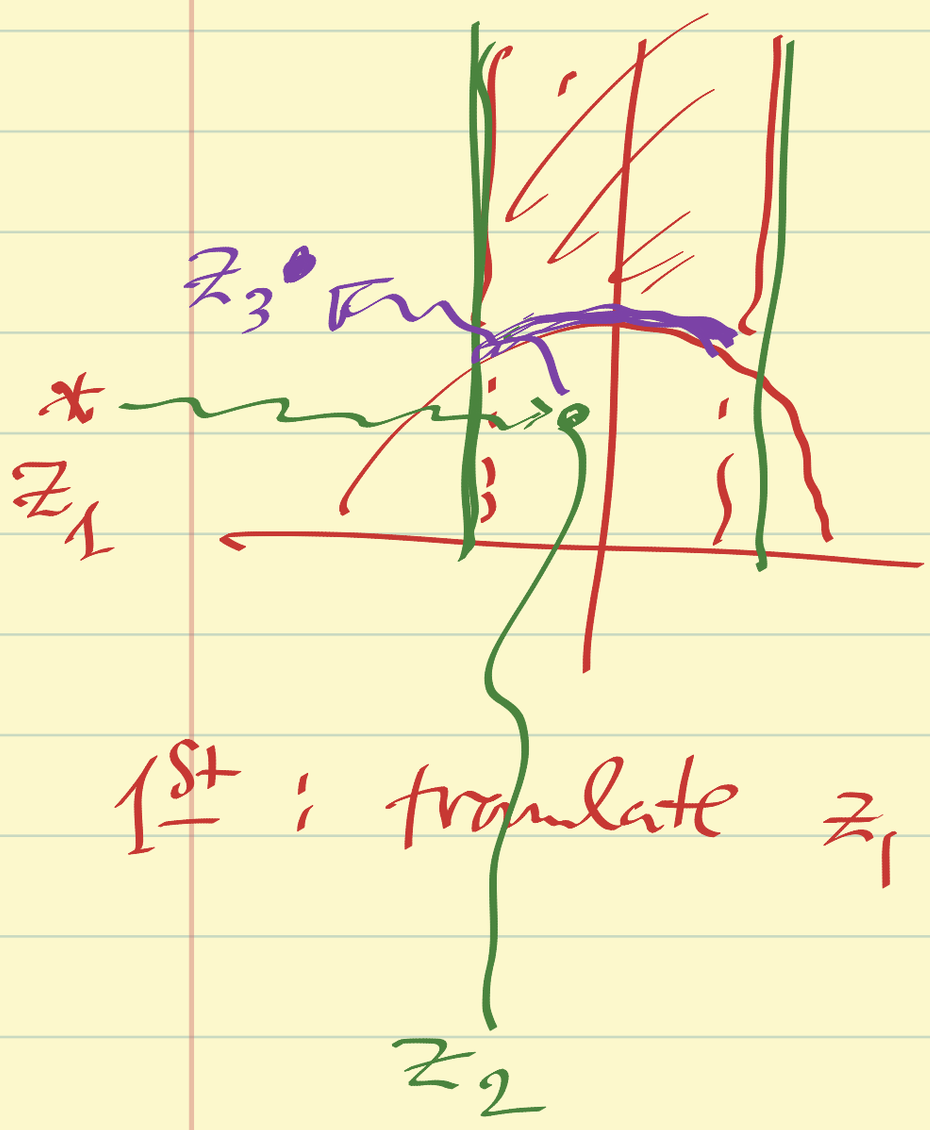
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(z) = z+1$

by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z) = -1/z$



"The algo" to move $z \in \mathcal{H}$ to fund. dmn:




repeatedly $z \longrightarrow z+n$, $n \in \mathbb{Z}$

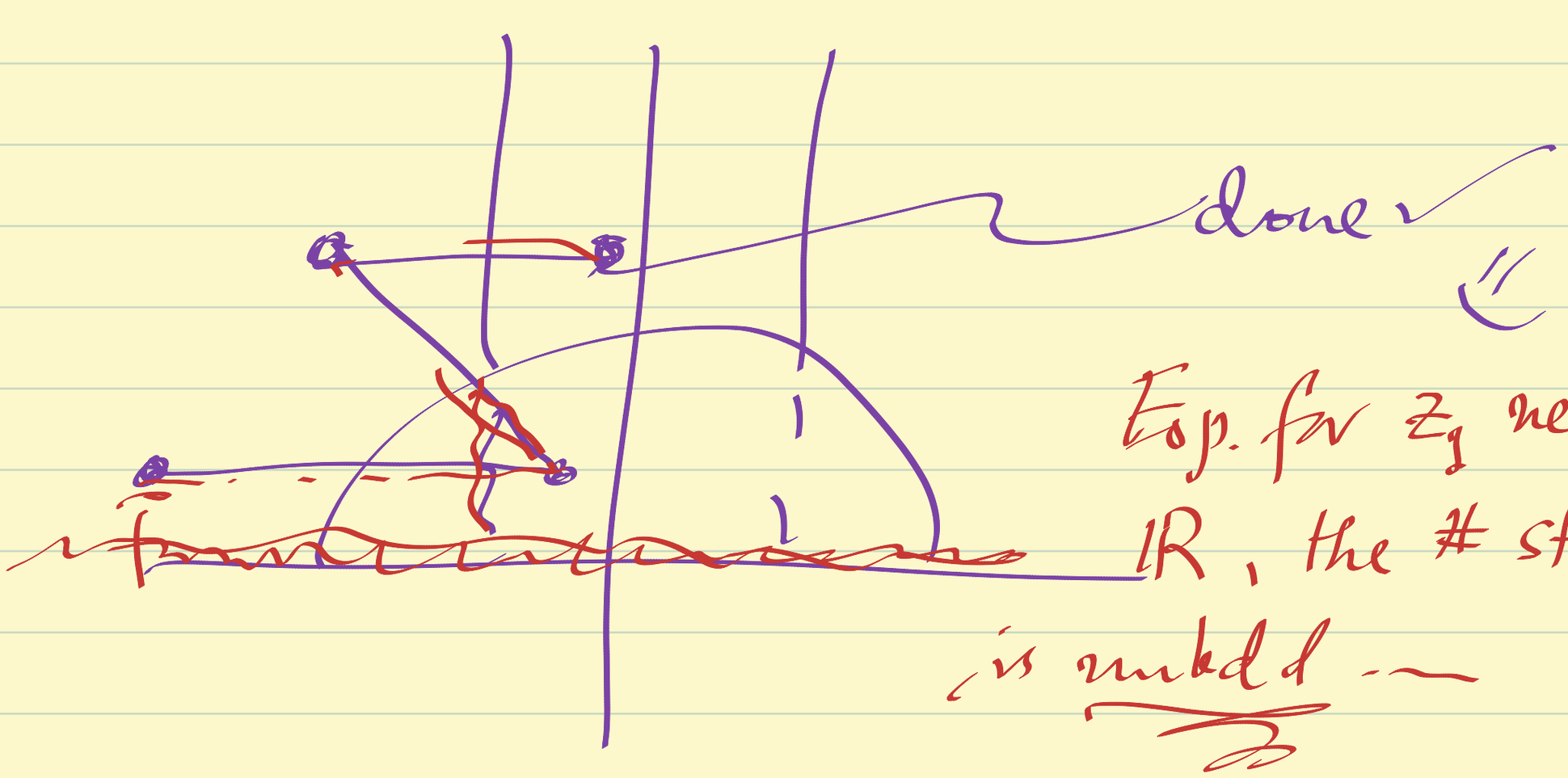
&/or invert $z \longrightarrow -1/z$

1st: translate $z_1$ to $z_2 \left(= z_1 + \exists n\right)$ in strip

$-1/2 \leq \mathrm{Re}(z) \leq 1/2$ ✓

2nd If $|z_2| < 1$ (or $\leq$), $z_3 = -1/z_2$ ✓

In pic: translate $z_3$ back to the strip:




Esp. for $z_1$ near $\mathbb{R}$, the # steps is unbdd —

Complexity $\cong$ Euclidean alg : $\exists$ bd $\nearrow$

$\Gamma\backslash\mathfrak{h}$ $\qquad$ $\pm$ geometry of $\Gamma\backslash\mathfrak{h}$

§5 – Fcns on $\Gamma\backslash\mathfrak{h}$ $\qquad$ $(\mathfrak{h} \subseteq \mathbb{C})$

$\|$

holo $\qquad$ Thm Use Arg principle to show that

$\|$

mero

for holo $f$ on $\Gamma\backslash\mathfrak{h}$,

$\|$

$f$ on $\mathfrak{h}$

& $\Gamma$-invt

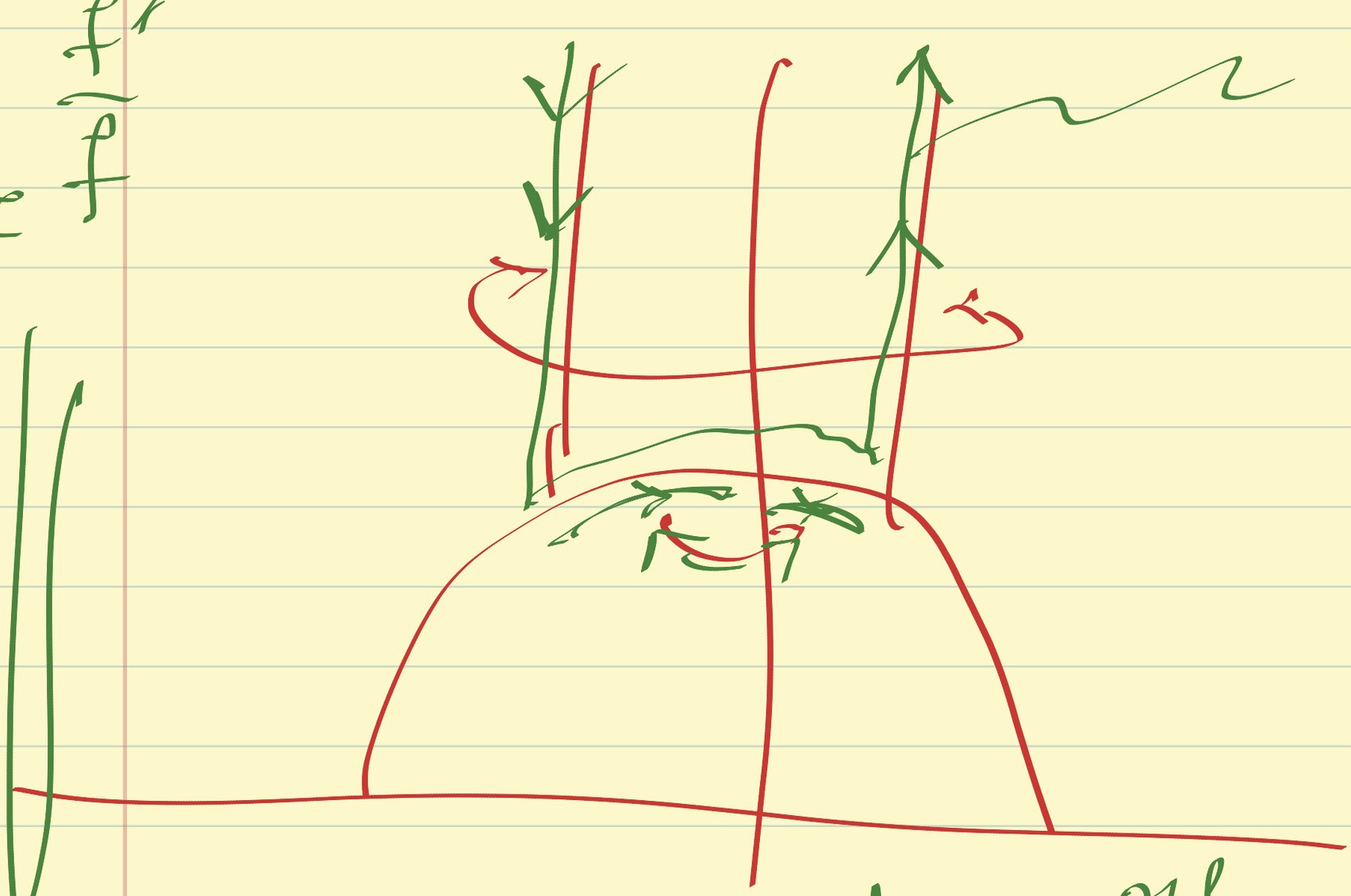
cube root

$\|$

$$\frac{\text{order van. @ } i}{2} + \frac{\text{ord van. @ } \omega}{3} + \sum_{\text{other } z_0 \in \text{fun. dm}} \text{order van @ } z_0 + \underbrace{\text{order of "@ } \infty\text{"}}_{\text{clarify!?!}} = 0$$

Lof!

$\int_{\partial \mathcal{F}} \frac{f'}{f}$



$\mathcal{F}$ = fun dmn

$2\pi i \times (\#\,0's - \#\,\infty's)$

Counts mults ✓

Oth, by invce,

$f(z+1) = f(z)$

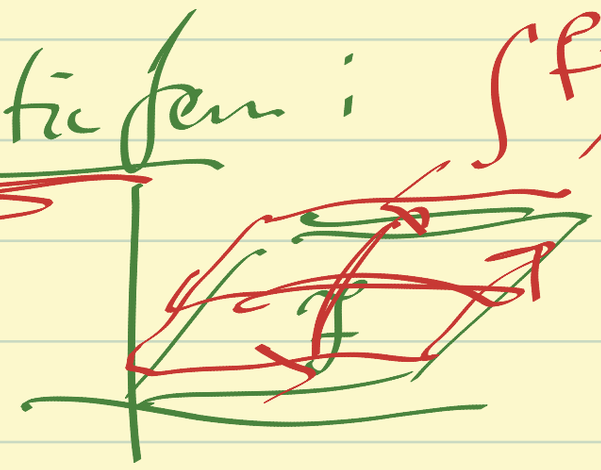
$\Downarrow$

verticals cancel !

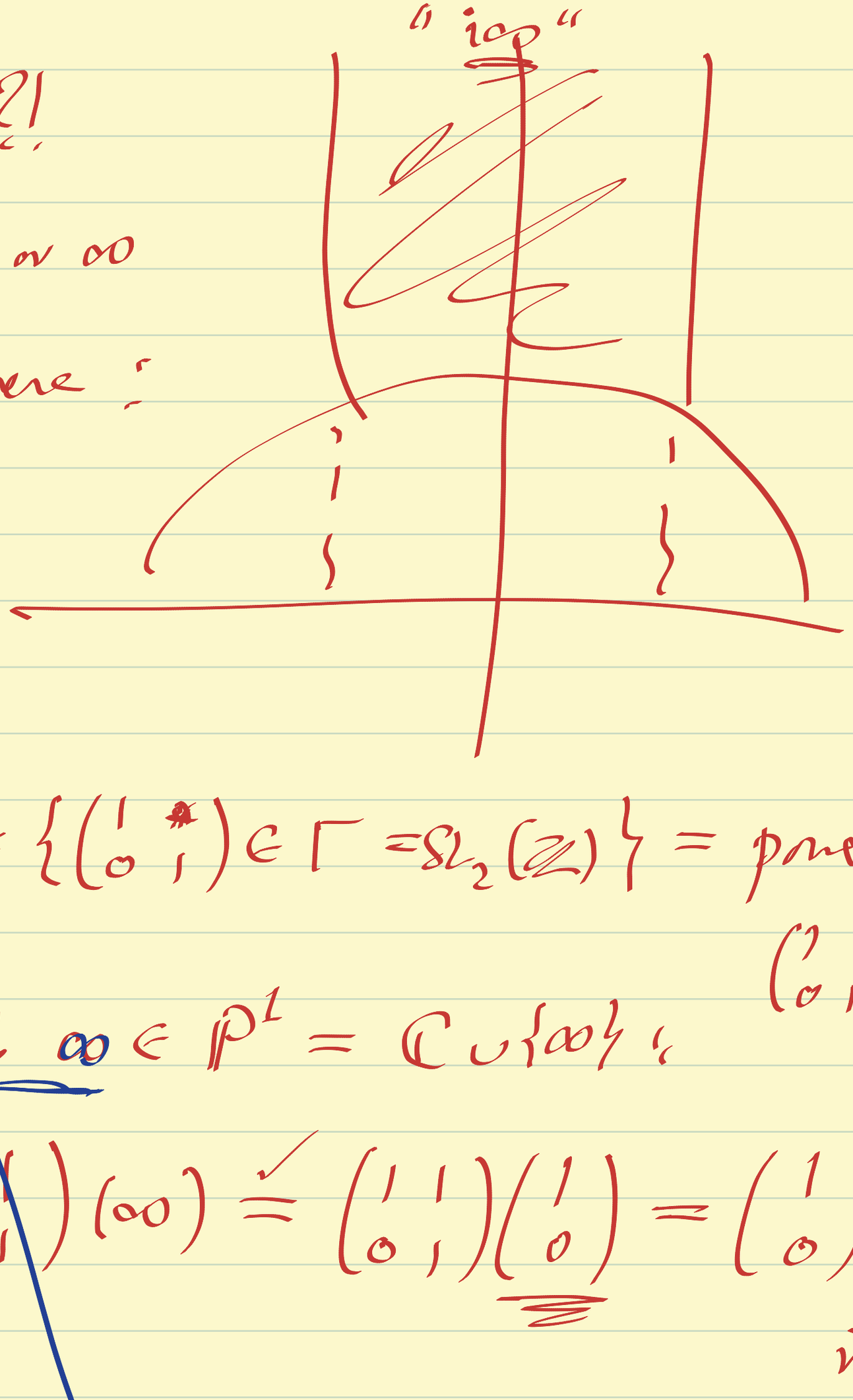
$\therefore$ sim, $\int_{\text{little arc}} \equiv 0$

$\equiv$ is $0$ ✓ "

∃ analogous fact for elliptic fcn: $\int f'/f$



At $\infty$ ?!




Order of 0 or $\infty$

of f there:

Let $\Gamma_\infty = \{\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \in \Gamma = SL_2(\mathbb{Z})\}$ = powers of $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

$\Gamma_\infty$ stab $\infty \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$;

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}(\infty) \overset{\checkmark}{=} \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} = \infty$$

in homog coords

logic

+ intuitive: "$\frac{1\cdot\infty + 1}{1} = \infty$"

abuse-able

Obv. heuristic is correct ✓"

notation for isotropy gp: $G$ act on $S$,

then $G_{s_0} = \{g \in G : g(s_0) = s_0\}$

**Claim**: holo $f$ on $\mathbb{H}$ (not nec. holo "@ $\infty$")

has "Fourier expn" $\rightsquigarrow$ $q$-expansion !!

$$f(x+iy) = \sum_{n\in\mathbb{Z}} \underbrace{c_n}_{\in\mathbb{C}} e^{2\pi i n z}$$

**pf**: $f$ is periodic in $x$: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}(x+iy)$

$= (x+1)+iy$

$\hookrightarrow$ $f \in C^\infty$, $f(x+iy) = \sum_{n\in\mathbb{Z}} c_n(y) \cdot e^{2\pi i n x}$

$\underline{\quad\quad}$ F-expn in $x$

+ holo: $c_n(y) = {}^\exists c_n \cdot e^{-2\pi n y}$

So $c_n(y) = c_n \cdot e^{-2\pi n y} \cdot e^{2\pi i n x}$

$= c_n \cdot e^{2\pi i n z}$