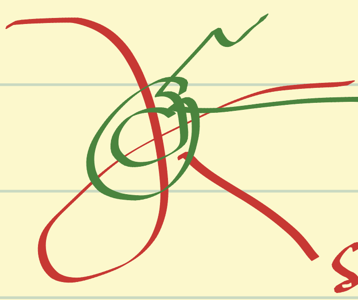
"Blowing-ups (points)" 02 May 2022

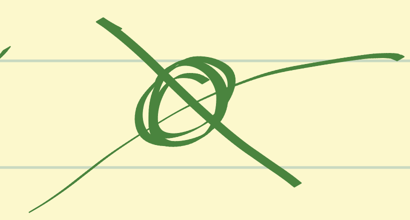
≡ (small case(s) of) "resolution of singularities"

(e.g., purpose of "smoothing out" is to be able to correctly apply (!?!) that is correct for smooth curves/surfaces/—

very Old idea:



locally ≃



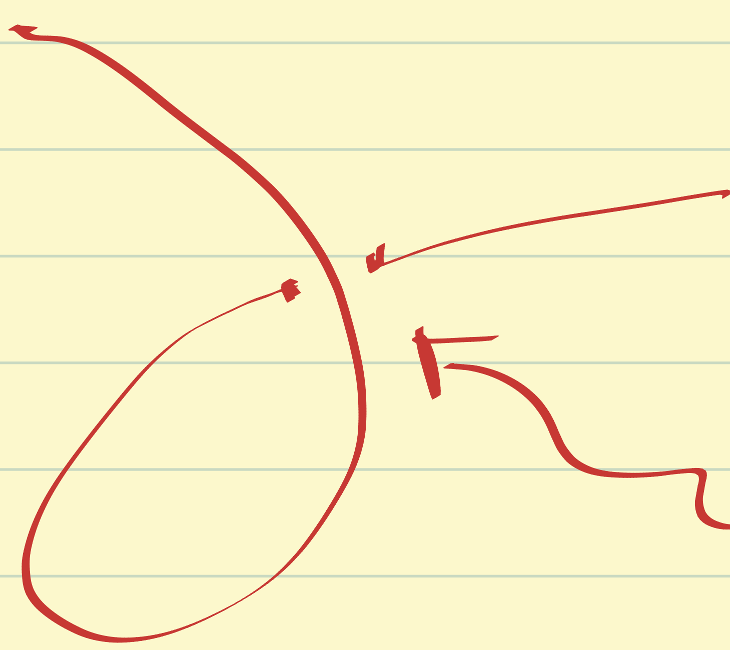
not smooth

self-intersection

⟿ miscounting ?!

desingularize: pull apart those "branches" of curve

easy: mash it down "



one curve goes under the other, not intersecting

"Resolution of sings":

known for curves over $\mathbb{C}$ or alg. closed char 0

140+ years!?!

200?!

Success! slightly mangled curves can be smoothed-out! ✓ :-)

1960's Abhyankar: surfaces & some 3-folds in arb. char (?!)

$\exists$ algorithms!?!

(prior to Gröbner)

1966 H. Hironaka: general case, char = 0!

1990's De Jong "alterations" ✓

Tangibly

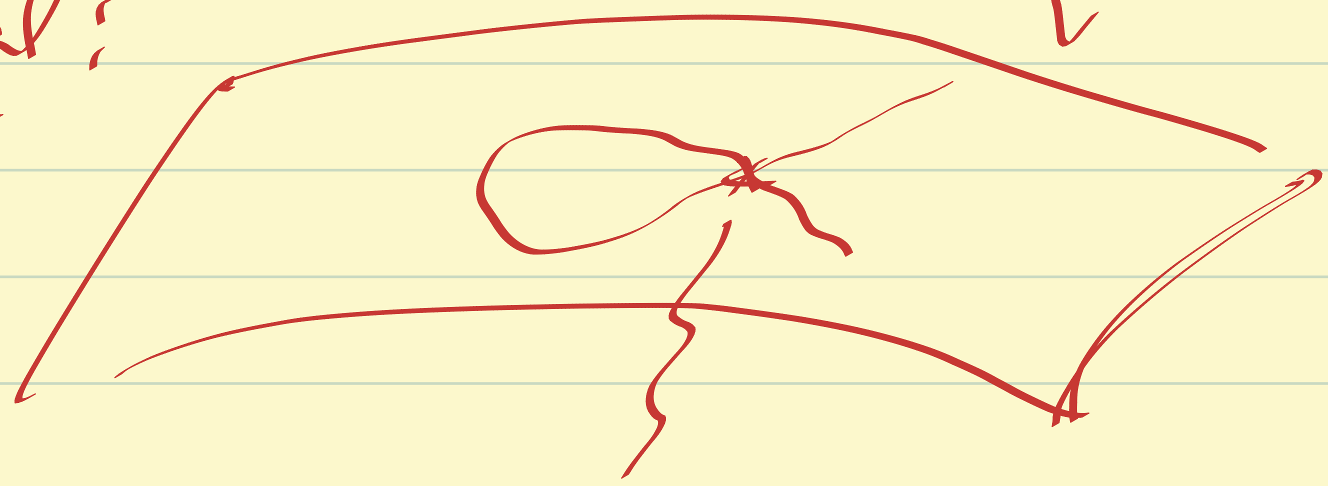
§ Formulaic aspect of $\alpha \rightsquigarrow \alpha$

(Italian school of alg. geom.) in 19th C. ✓

0th (Process) Blow up a point in $\mathbb{P}^n$ (eg., $\mathbb{P}^2$)

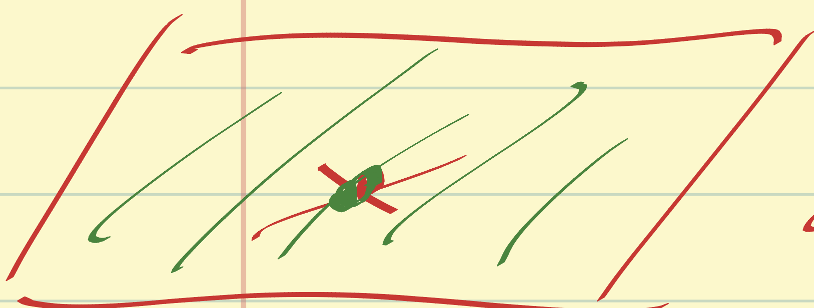
$\mathbb{C}^n$

Pictorial:

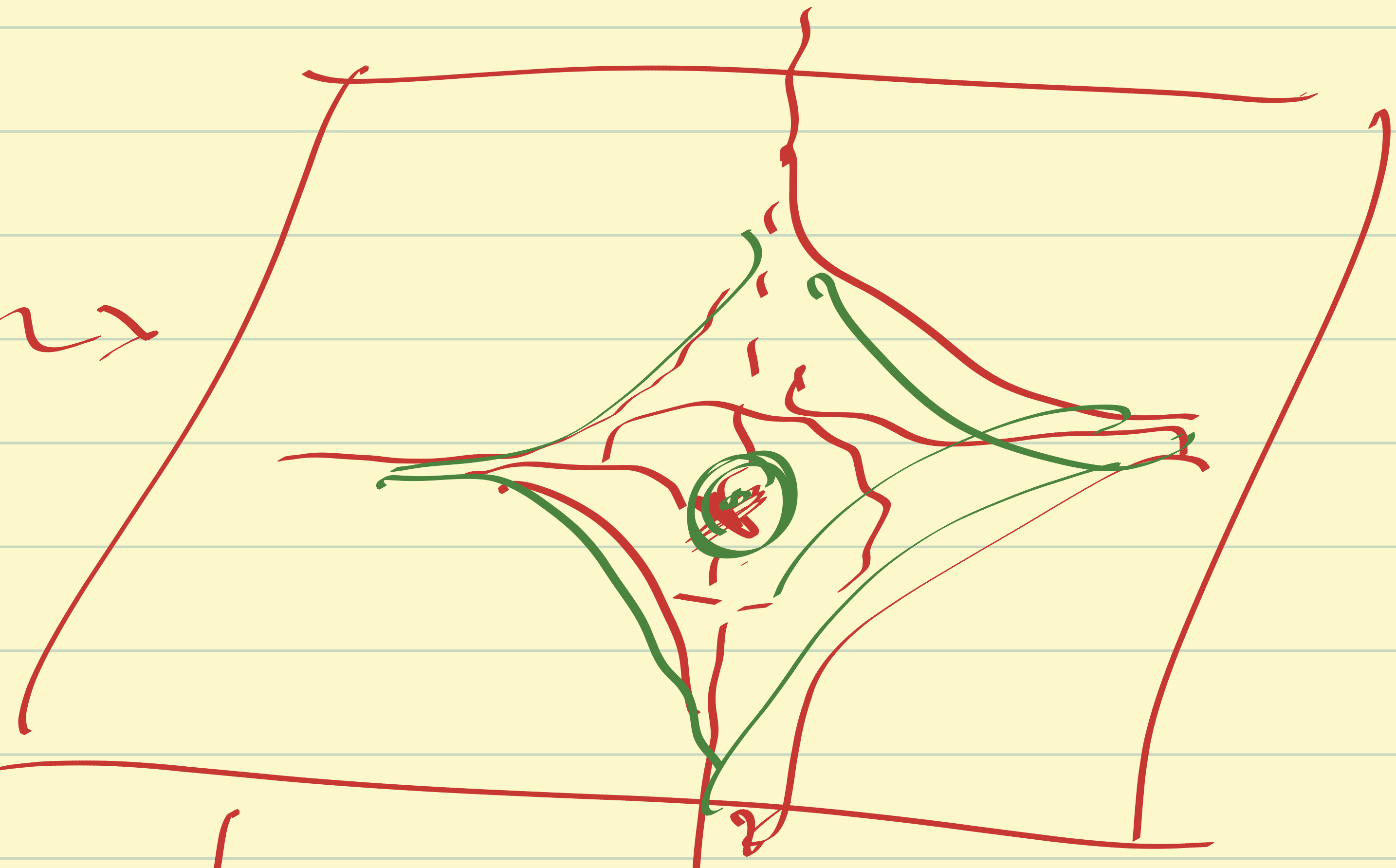


$x_0$

"Replace $x_0$ by all tangent dirs @ $x_0$" !

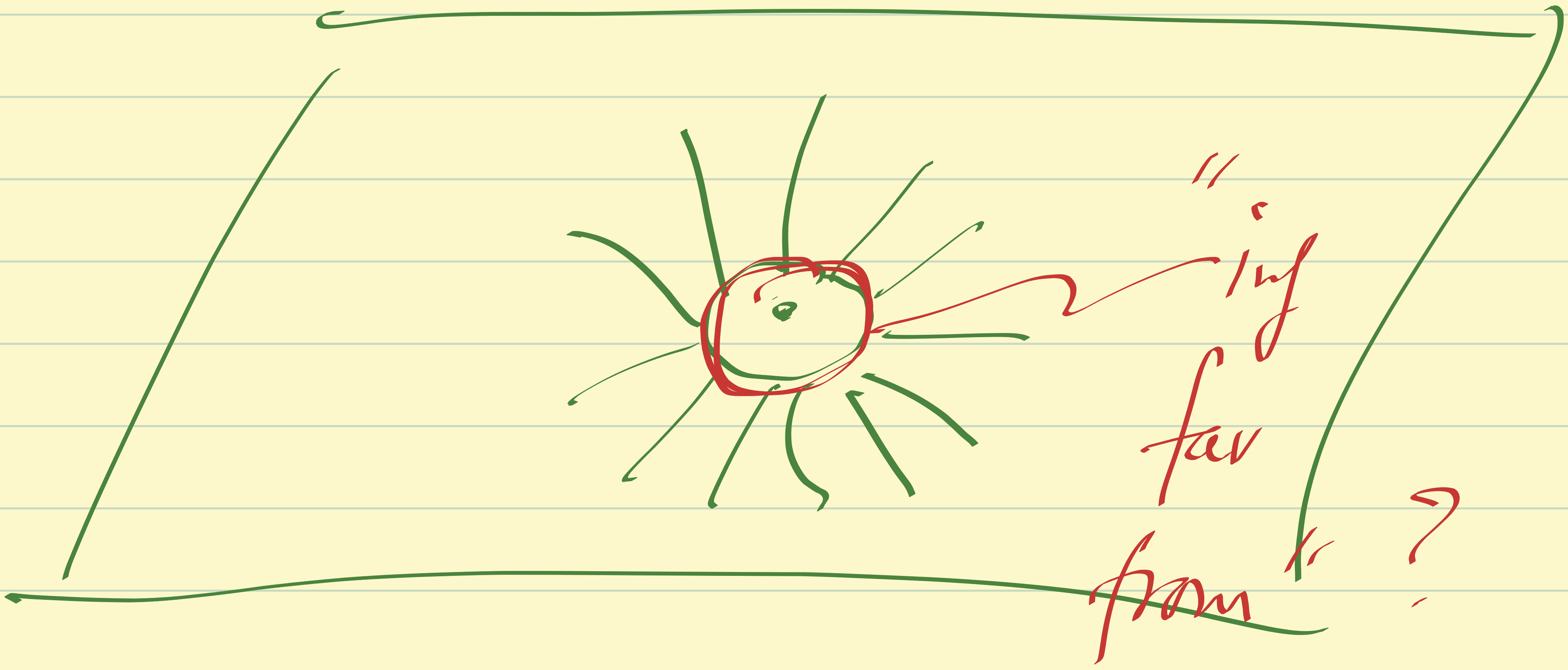


$\rightsquigarrow$ ?!



pt $\rightsquigarrow$ circle ✓

of direction! ✓ "




rays going to it.

What is the map (= blow up)

$$x_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{P}^n$$

($Gl_{n+1}$ transitive on $\mathbb{P}^n$ so except irrelevant)

Map $\underbrace{\boxed{\mathbb{P}^n - \{0\}}}_{\substack{\ni \\ x_0}} \longrightarrow \left(\underbrace{\mathbb{P}^n - \{0\}}_{\substack{\not\ni \\ x_0}}\right) \times \mathbb{P}^{n-1}$

$\mathbb{C}^n$ $\qquad$ $\mathbb{C}^n$

$\mathbb{P}^{n-1}$ = tangent directions @ $x_0$

h

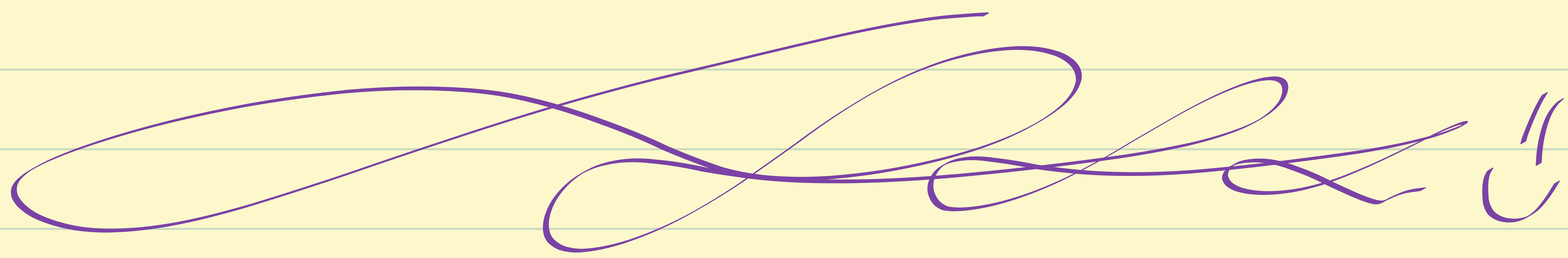
$$\text{"}x\text{"} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \times \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad ?!$$

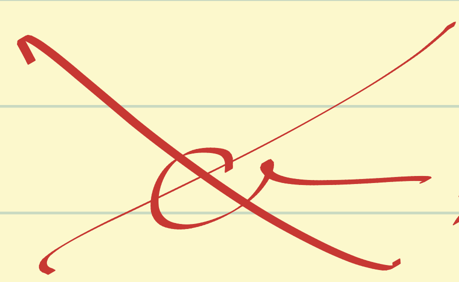
$\neq$

not all $x_j$'s $= 0$

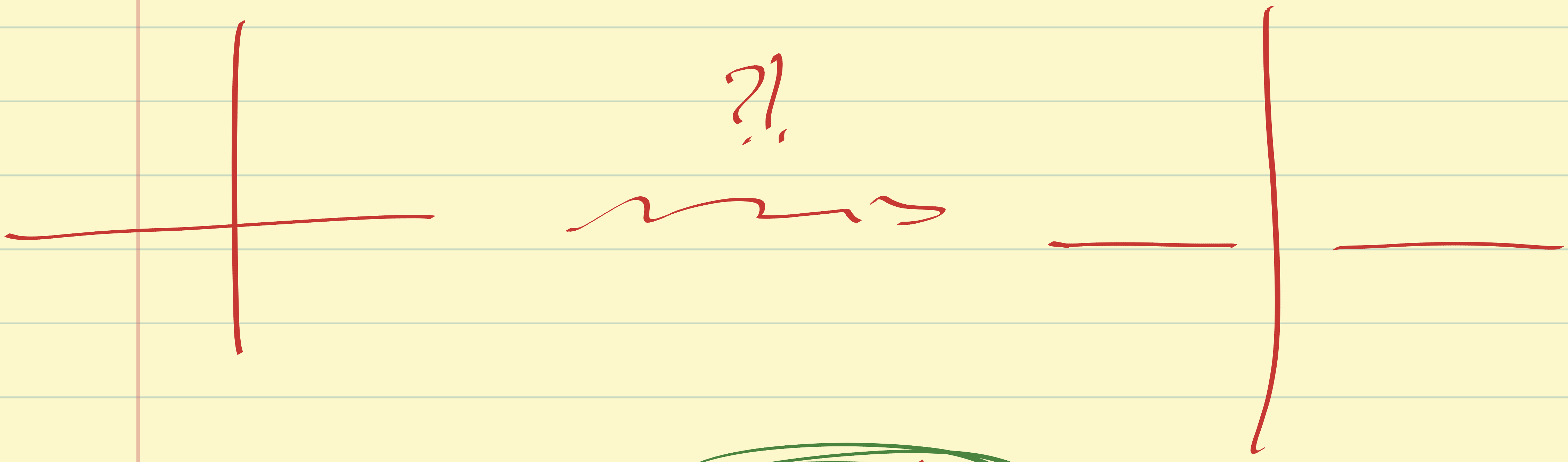
<u>Def'n</u>: Blow-up of $\mathbb{P}^n$ (or $\mathbb{C}^n$) at $x_0$ is <u>closure</u> (top !?! $\exists$ choices: here use std top on $\mathbb{C}$ &/or $\mathbb{R}$) of <u>image</u> of $\mathbb{P}^n - \{x_0\}$ (or $\mathbb{C}^n - \{x_0\}$, —) in

$$\underbrace{\left(\mathbb{P}^n - \{0\}\right)}_{\text{physical space}} \times \underbrace{\mathbb{P}^{n-1}}_{\text{directions @ } x_0}$$



<u>Effect</u> on  actually  !?!

$$\underbrace{xy + x^n + y^n = 0}_{\text{unhelpful —}} \qquad \boxed{xy = 0} \text{ in } \mathbb{R}^2$$




$$C = \{(x,y)\in\mathbb{R}^2 : xy = 0\}$$

$$= \{(t,0) : t\in\mathbb{R}\} \cup \{(0,t) : t\in\mathbb{R}\}$$

Blow-up map at $(0,0)\in\mathbb{R}^2$

not both 0

$$\begin{pmatrix}x\\y\\1\end{pmatrix} \longrightarrow \begin{pmatrix}x\\y\\1\end{pmatrix}\times\underbrace{\begin{pmatrix}x\\y\end{pmatrix}}_{\text{either}}$$

$$\begin{pmatrix}x\\y\\1\end{pmatrix} \in \mathbb{R}^2 \subset \mathbb{RP}^2$$

either $= \begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{RP}^1$

on one line

$P_0$

$$\begin{pmatrix}\text{large } x\\ 0\\ 1\end{pmatrix} \longrightarrow \begin{pmatrix}x\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\0\end{pmatrix}$$

seps pts !!

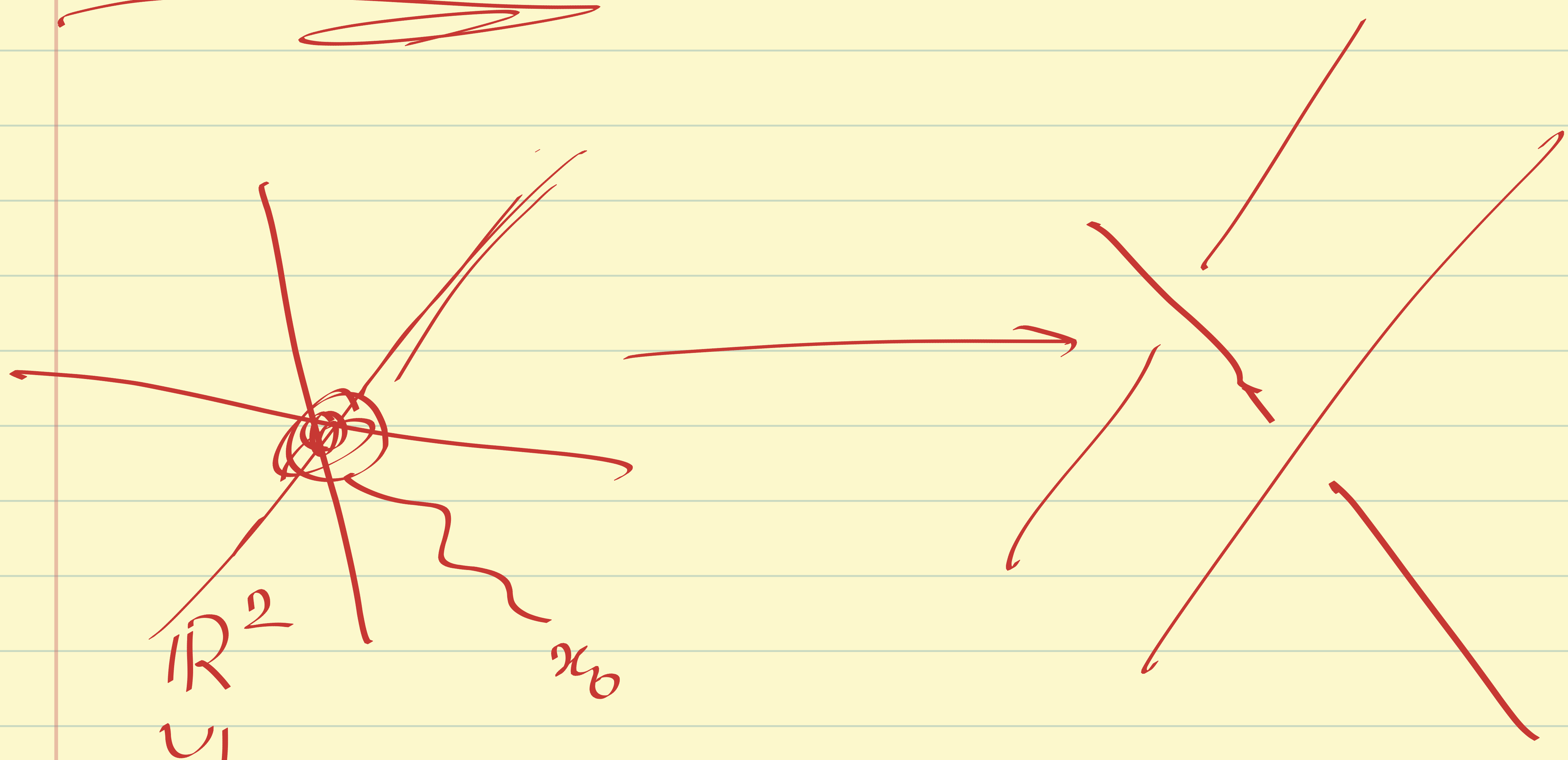
& vice-versa

Again: blow up $x_0 \in \mathbb{C}^n$ :

map $\quad \mathbb{C}^n - \{x_0\} \longrightarrow \left(\mathbb{C}^n - \{x_0\}\right) \times \underbrace{\mathbb{P}^{n-1}}_{\text{directions @ } x_0}$

take closure of the image of

## For that matter :




$$\mathbb{R}^2 \ni \begin{pmatrix} at+b \\ ct+d \end{pmatrix} \longrightarrow \begin{pmatrix} '' \end{pmatrix} \times \text{one of 3 pts in } \mathbb{P}^1$$

~ which line

Can repeat blow-up of pts
&/or do higher-order

Instead $\mathbb{C}^n - x_0 \longrightarrow (\mathbb{C}^n - x_0) \times$ directions at $x_0$

Could " $\longrightarrow$ " $\times$ $2^{nd}$-order !

$$\lambda \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow x \times \left\{ \begin{matrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1 x_3 \\ \vdots \end{matrix} \right. \quad (\text{mean } 0)$$

$$\in \mathbb{C}^n - \{x_0\}$$

Eg., "cusp"

$\binom{n}{2} - 1$

projective