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Holomorphy of compact integrals of holomorphic

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Let $f(t,z)$ be a function on $[a,b]\times U$ for a finite interval $[a,b]\subset\mathbb{R}$ and non-empty open $U\subset\mathbb{C}$.

[0.1] Claim: Suppose that $f(t,z)$ is jointly continuous, as a function of two variables, and that for each fixed t is holomorphic in z. Then $F(z)=\int_a^b f(t,z)\,dt$ is holomorphic in z. Further, if $\frac{\partial}{\partial z}f(t,z)$ is also jointly continuous, then the natural differentiation property holds:

$$F'(z) \;=\; \frac{d}{dz}\int_a^b f(t,z)\,dt \;=\; \int_a^b \frac{\partial}{\partial z}f(t,z)\,dt$$

Proof: Apply Cauchy's formula to $f(t,z)$ for each fixed t:

$$F(z) \;=\; \int_a^b f(t,z)\,dt \;=\; \int_a^b \Big(\frac{1}{2\pi i}\int_\gamma \frac{f(t,w)}{w-z}\,dw\Big)\,dt$$

where γ traces a circle (contractible in U) with z in its interior. Let $\gamma:[c,d]\to\mathbb{C}$ be any particular parametrization, so

$$\int_a^b \Big(\frac{1}{2\pi i}\int_\gamma \frac{f(t,w)}{w-z}\,dw\Big)\,dt \;=\; \frac{1}{2\pi i}\int_{[a,b]\times[c,d]} f(t,\gamma(\theta))\cdot\gamma'(\theta)\cdot\frac{1}{\gamma(\theta)-z}\,dt\,d\theta$$

Direct estimates will show that this double integral is complex differentiable in z. Certainly

$$\frac{1}{w-(z+h)}-\frac{1}{w-z} \;=\; \frac{h}{(w-(z+h))(w-z)}$$

Thus,

$$\frac{F(z+h)-F(z)}{h} \;=\; \int_{[a,b]\times[c,d]} f(t,\gamma(\theta))\cdot\gamma'(\theta)\cdot\frac{1}{\gamma(\theta)-z}\cdot\frac{1}{\gamma(\theta)-(z+h)}\,dt\,d\theta$$

For each fixed z inside γ, we show that this double integral has a limit as $h\to 0$. First, for fixed z inside γ, the continuous function $(t,\theta)\to f(t,\gamma(\theta)\cdot\gamma'(\theta)/(\gamma(\theta)-z)$ is necessarily *uniformly* continuous on $[a,b]\times[c,d]$, because $[a,b]\times[c,d]$ is *compact*. Thus, it suffices to show that, for fixed z inside γ

$$\frac{1}{\gamma(\theta)-(z+h)} \;\longrightarrow\; \frac{1}{\gamma(\theta)-z}$$

uniformly in θ. This is true for general reasons, but, also, again, explicitly,

$$\frac{1}{w-(z+h)}-\frac{1}{w-z} \;=\; \frac{h}{(w-(z+h))(w-z)}$$

Certainly $|w-z|=|\gamma(\theta)-z|$ has a positive inf μ. Further, we can take h small enough so that $z+h$ is closer to z than to any $w=\gamma(\theta)$. Thus, $|w-(z+h)|\geq\frac{1}{2}\cdot|w-z|$. Thus, we have a uniform estimate

$$\Big|\frac{1}{w-(z+h)}-\frac{1}{w-z}\Big| \;\leq\; \frac{|h|}{\frac{1}{2}\cdot\mu\cdot\mu}$$

Thus, the integrand goes to

$$f(t,\gamma(\theta))\cdot\gamma'(\theta)\cdot\frac{1}{(\gamma(\theta)-z)^2}$$

uniformly in t, θ.

The Riemann integral has the property that the integral on a compact set of a uniform limit of continuous functions is the integral of the limit. Thus, the limit exists, and $F(z)$ is complex-differentiable.

Further, from

$$F'(z) \;=\; \frac{1}{2\pi} \int_{[a,b]\times[c,d]} f(t,\gamma(\theta)) \cdot \gamma'(\theta) \cdot \frac{1}{(\gamma(\theta)-z)^2} \; dt\, d\theta$$

integrating by parts in θ (since γ is a closed curve),

$$F'(z) \;=\; \frac{1}{2\pi i} \int_{[a,b]\times[c,d]} \gamma'(\theta) \cdot \frac{\partial}{\partial w} f(t,\gamma(\theta)) \cdot \frac{1}{\gamma(\theta)-z} \; dt\, d\theta$$

$$=\; \frac{1}{2\pi i} \int_{[a,b]} \int_{\gamma} \frac{\partial}{\partial w} f(t,w)\, \frac{1}{w-z} \; dw\, dt \;=\; \int_{[a,b]} \frac{\partial}{\partial z} f(t,z)\, dt$$

by Cauchy's formula. This is the desired differentiation-under-the-integral formula. ///
